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# Chapter 5:

## Making it Manifest: The Intellectual Value of Good Variables

### 5.1 Hidden Symmetries and Manifest Properties

When discussing the symmetries of models, physicists and chemists sometimes speak of "hidden symmetries." These are symmetries of the model that certain choices of variables obscure. A system possesses a hidden symmetry when its full symmetry group is larger than its "apparent" or "obvious" symmetry group. Paradigmatic examples include the classical and quantum two-body problems (which have a hidden hyperspherical symmetry) and the isotropic harmonic oscillator (which has a hidden special unitary symmetry). A more recent example occurs in the context of $\mathcal{N} = 4$ super Yang–Mills theory, whose tree-level amplitudes possesses a hidden dual superconformal symmetry, along with a larger hidden symmetry known as the Yangian.

By reformulating these models, physicists were able to make these hidden symmetries manifest. The process of making a symmetry manifest distinguishes hidden symmetries from their "obvious" counterparts: non-hidden symmetries were already made manifest in a prior formulation. In some cases, a symmetry is manifest because it is "worn on the sleeves" of a relevant expression. As we will see, this notion of *wearing a property on the sleeves* is a special case of making a property manifest.

The phenomena of manifest symmetries suggests a problem for conceptualism that fundamentalism avoids. *Prima facie*, it is intellectually significant to make a hidden symmetry manifest.[1] It does not seem to be merely a convenient re-expression of a theory or model's known properties. Yet, it is initially not clear how conceptualism can accommodate the intellectual significance of making a symmetry manifest. This is because there

[1]Recall that 'intellectual significance' is a non-practical dimension of epistemic significance.

is typically a translation procedure between variables that obscure a symmetry and variables that make this symmetry manifest. Hence, it initially seems that both sets of variables must express the same set of epistemic dependence relations. If this were so, then conceptualism would fail to save the intuition that something of intellectual importance can occur when scientists make a symmetry—or other property—manifest.

In contrast, fundamentalism suggests a simple account of the intellectual significance of making properties manifest. Expressive means that make more fundamental properties manifest carve nature more closely at the joints. Insofar as a symmetry qualifies as fundamental, making it manifest would likewise count as being intellectually significant. Indeed, symmetries are connected with physical invariants, and invariants are typically taken to be physically fundamental. If conceptualism cannot provide a satisfying account of making symmetries manifest, it would seem as though fundamentalism has the upper hand in this context. This chapter provides a conceptualist account of the significance of making properties manifest, including symmetries. In keeping with the methodological desiderata of Chapter 1, I will not appeal to ontologically-primitive differences in joint-carving or fundamentality. Such differences might obtain, but I will remain agnostic as to whether they do. Instead, I will locate a source of non-practical, epistemic value in making properties manifest.

Section 5.2 begins with a general account of what it means for a fact to be manifest rather than hidden. I then consider the ubiquitous phenomena of expressions that wear a property "on the sleeves." Section 5.2.1 analyzes this as a special case of making a property manifest. I illustrate my account with simple examples from math, physics, and logic. Next, Section 5.3 applies my account to a simple example from language translation: some languages make the meaning of a word more manifest than others. Section 5.5 considers the more complicated but still prosaic context of coordinate transformations.

In all of these cases, conceptualism threatens to either collapse into instrumentalism or risk expanding into fundamentalism. For instance, if no coordinate choice carves the system more closely at its joints, then how can we intellectually privilege one set of coordinates over another? For many coordinate transformations, there seems to be nothing but convenience to decide between them. Section 5.4 responds to these worries by clarifying the non-practical epistemic value of making properties manifest. Making a property manifest is valuable whenever it rules out epistemically possible solutions to a given problem. This ruling out of possibilities has epistemic value independently of any practical value. Since I do not appeal to primitive differences in fundamentality, my account shows that fundamentalism is not needed even in this context. My argument complements Woodward's (2016, p. 1056) argument that appeals to joint-carving do not help us resolve philosophical problems about good variable choice.

Nevertheless, a fundamentalist might object that my account fails to preserve ordinary judgments regarding relative fundamentality. Physicists and mathematicians commonly view some variable choices as being more fundamental or deeper than others. Fundamentalism seems well-suited to vindicate these ordinary judgments of fundamentality. In contrast, conceptualism faces the burden of accounting for them without appealing to substantial metaphysical commitments. To meet this burden, Section 5.6 proposes an expressivist account of fundamentality. To judge that a formulation X is more fundamental than a formulation Y is to express a mental state of *being for privileging X over Y*. Using the example of gauge choices in quantum field theory, Section 5.7 develops a separate argument against fundamentalism. Making one fundamental property manifest often comes at the cost of obscuring others. This provides some reason to be pessimistic that physics will ever arrive at a fundamental language that avoids these trade-offs.

I end by considering examples that have motivated the entire enterprise: hidden symmetries. Section 5.8 illustrates my framework in the context of the hidden hyperspherical symmetry of the nonrelativistic hydrogen atom. In many formulations of the hydrogen atom, this symmetry is hidden while hydrogen's spherical symmetry is manifest. By moving to momentum space, we can make this hidden $SO(4)$ symmetry manifest. Finally, Section 5.9 considers hidden symmetries in the context of $\mathcal{N}=4$ super Yang–Mills theory. I describe the chain of variable choices that allow us to make a hidden dual superconformal $SU(2,2|4)$ symmetry manifest. At each step in this long chain of variable changes, we acquire intellectually significant benefits.

## 5.2 Manifest vs. Hidden Facts

To account for the wide variety of cases that interest me, I propose the following account of manifest facts: a fact is *manifest* at a given stage in a problem-solving plan provided that an agent who implements that plan ought to infer that fact. More precisely:

> **Manifest fact**: a fact $F$ is manifest in epistemic circumstance $C$ provided that an agent in state $C$ ought to infer that the fact $F$ obtains.

On this characterization, solutions are always manifest at the end of a successful problem-solving plan: an agent that implements the plan ought to infer the solution. I take this feature to be a conceptual requirement of any definition of 'manifest fact.' It is constitutive of a successful problem-solving plan that it makes the solution manifest. Otherwise, the plan has not reached its aim and to that extent remains unsuccessful.

By 'agents,' I mean to include both sapient and non-sapient problem-solvers, such as algorithms implemented by a computer program. Sapient agents have a further capacity for grasping a problem-solving plan, thereby understanding it in a psychological sense. Sapient agents can not only implement a plan but also understand it.

My characterization of manifest facts treats it as a normative aspect of problem-solving plans. Whether or not a fact is manifest depends on what we epistemically ought to infer. Some may be wary of normativity, but there is nothing to fear, even for a hard-nosed empiricist or naturalist like myself. Gibbard (2012) provides an ontologically non-mysterious account of what constitutes these ought-claims. They simply amount to plans for action or belief. In particular, "epistemic ought beliefs amount to plans for degrees of credence" (2012, p. 178). To simplify the discussion, I will typically talk in terms of full-belief, although it is straightforward to generalize the account to degrees of credence. Degrees of credence accommodate problem-solving plans that involve inductive rather than deductive reasoning.

We can likewise characterize what it means for a fact not to be manifest, i.e. to be non-manifest. We simply negate the characterization of a manifest fact:

> **Non-manifest fact**: a fact $F$ is not manifest in epistemic circumstance $C$ provided that it is not the case that an agent in state $C$ ought to infer $F$.

For instance, solutions are not manifest at the beginning of problem-solving (otherwise, one would not need to engage in problem-solving). It is not the case that one ought to infer the solution to a problem before carrying out an adequate problem-solving plan.

That a fact is not manifest does not necessarily entail that it is hidden or obscured. It may sometimes be permissible for an agent to infer a fact that is not manifest. To characterize what it means for a fact to be hidden, I propose the following logically stronger definition:

> **Hidden fact**: a fact $F$ is hidden in epistemic circumstance $C$ provided that it is *impermissible* for an agent in state $C$ to infer that $F$ obtains.

Equivalently, a fact is hidden provided that an agent ought not infer it.

Epistemic-ought claims play an important role in my account of manifest, non-manifest, and hidden facts. But what does it mean to say that an agent in a particular circumstance *ought to infer* a given fact? We can gloss this as follows: if an agent ought to infer $F$, but they fail to infer $F$, then their inferential omission warrants disapproval. This disapproval is of a specifically epistemic variety: it is disapproval on epistemic grounds. In the cases I consider, it involves disapproval of the agent's subsequent epistemic state.[2]

If an agent fails to infer the correct answer, then they either (i) infer an incorrect answer, (ii) fail to realize that they know how to solve the problem (e.g. by falsely believing that they do not have enough information), or (iii) simply do not know how to solve the problem. The first two cases involve a kind of epistemic mistake: the agent believes something false (either the wrong answer or an erroneous belief about what is possible). In the third case, the agent displays an epistemic deficiency: they are unable to implement an appropriate problem-solving plan. Of course, this third case warrants disapproval only if the agent ought to know better, i.e. ought to be able to implement the plan. In the cases I consider, I will assume that the agent either knows or ought to know how to implement such a plan. A computer program can malfunction in all three of these different ways. It might halt at the wrong answer, fail to halt when it should, or simply stop working entirely (and not because it has been turned off!).

A simple example from graph theory illustrates the various components of my account. Given a graph (i.e. a collection of edges and vertices), one general question is whether the graph has the property of planarity. Planar graphs admit a representation such that no edges cross in the plane. For any given planar graph, most of its representations hide the fact that it is planar. These representations hide the planarity of the graph by representing two or more edges as crossing. In contrast, other representations demonstrate that a planar embedding is possible: they make manifest the planarity of the graph.[3]

[2] In general, we might also epistemically disapprove of an agent's epistemic process. For instance, we disapprove of an agent who gets the right answer but for the wrong reasons, such as by accidental cancellation of two compensating mistakes.

[3] Elsewhere in pure mathematics, number theory provides numerous cases where reformulating a problem makes otherwise hidden patterns manifest (Ash and Gross 2008).

If a student of graph theory is shown a planar representation of a graph, the student ought to infer that the graph is planar. If they do not make this inference—drawing some other inference instead—then their inference warrants epistemic disapproval. For they have either i) inferred that the graph is not planar, ii) inferred that there is not enough information to solve the problem, or iii) realized that they don't know how to solve the problem. In the first two cases, they make an epistemic mistake. In the third case, they display a deficiency that they ought not have (given their background training in graph theory). They show that they lack sufficient understanding of graph theory, whereas they ought to have this understanding.

Of course, if it is not the case that an agent ought to have this background knowledge, then they make neither an epistemic mistake nor display an inexcusable epistemic deficiency. If you show a kindergartner a planar representation of a graph and ask them whether the graph is planar, they can permissibly reply that they have no idea what you are talking about. Although the kindergartner has an epistemic deficiency, they are excused from disapproval. It is not the case that they ought to understand graph theory. Likewise, if someone simply loses interest in solving a problem and walks away, we cannot epistemically disapprove of them for this. We might still, nonetheless, disapprove of their values and goals.

The definitions of manifest, non-manifest, and hidden facts reference an epistemic circumstance $C$. This circumstance encompasses both i) the background knowledge and capacities that the agent has and also ii) what information they are being presented with in a given problem-solving context. Sometimes, it will be convenient to isolate the latter information, calling it *the problem-specific epistemic circumstance* $P$. In the case above, both the graph theory student and the kindergartner are presented with the same problem-specific circumstance $P$, i.e. the same representation of the graph. But overall they are in different epistemic circumstances based on their different background knowledge. The planarity of the graph is manifest for the student of graph theory but not for the kindergartner.

The phenomena of perfect (or absolute) pitch helps illustrate why it is necessary to index what we ought to infer to our background knowledge and capacities. Consider two musicians presented with the same sound, such as a musical note sustained on a violin.[4]

[4]In virtue of being musicians, these agents understand the naming conventions linking frequencies to pitches.

The first musician has perfect pitch. In virtue of this, they ought to infer the pitch class of the note played, e.g. that it is a B-flat. To do this, they do not need any measuring device or even a reference pitch. The second musician does not have perfect pitch. Hence, it is not the case that they ought to infer that the note is a B-flat, just from hearing it. It is epistemically permissible for them not to know the pitch. In order for the pitch to become manifest to the second agent, they need a measuring device, such as a tuning fork, a digital tuner, or testimony. Using a digital tuner alters their epistemic circumstance, such that the pitch of the note becomes manifest. As described below, formulations that "wear a property on the sleeves" are analogous to having perfect pitch. They make it the case that one ought to infer the property without needing intermediary expressions, analogous to how someone with perfect pitch does not need an intermediary measuring device.[5]

The capacity of logical omniscience provides another illustration of how what an agent ought to infer can depend on their capacities. Logically omniscient agents ought to infer any logical consequence of a sentence or group of sentences. For them, all logical consequences are manifest. Clearly, this is not the case for us, in virtue of our lack of logical omniscience. As in Chapter 4, I am interested in agents that are not logically omniscient. Most of the epistemic differences that interest me here do not arise for logically omniscient agents. Unlike humans, such agents would have no reason to reformulate in many of the cases described below.

### 5.2.1 Simple examples, on the sleeves

My account of manifest facts leads straightforwardly to an account of what it means for an expression to wear a property "on its sleeves." I propose to understand this as follows:

> **To wear on the sleeves ('sleeve properties')**: a representation or expression $E$ wears a property $P$ on its sleeves provided there is a problem-solving plan that both
> (i) makes $P$ manifest and
> (ii) does so solely on the basis of manifest facts about $E$.

Unpacked, this definition comes to the following: applying an appropriate plan to the

[5]My account of manifest properties sheds light on Wittgenstein's discussion of what he calls "aspect-blindness" in Fragment xi of the *Philosophy of Psychology*. Wittgenstein is concerned with humans that lack "the ability to see something *as something*." He asks whether this "defect" would be comparable "to not having absolute pitch" and later answers that "aspect-blindness will be *akin* to the lack of a 'musical ear'" (2009 [1949], 224–225, §257, 260). In my terminology, aspect-blindness occurs when the changing of aspects—such as the Gestalt shift of the Necker cube—is not manifest to an agent.

expression $E$ makes the property $P$ manifest. Importantly, this plan must rely solely on properties of $E$ that are already manifest (before implementing the plan). Collectively, the expression and the plan generate an epistemic circumstance in which the property is manifest. Typically, these plans are built around a central epistemic dependence relation, which we exploit to make the property $P$ manifest. In this section, I illustrate my account using some simple examples from logic and physics.

### Sleeve Properties in Truth-Functional Logic

Sentential logic provides a wellspring of examples of "sleeve properties."[6] Among different but truth-functionally equivalent sentences, often one wears a property on the sleeves that another obscures. Much of the interest in certain kinds of normal forms for truth-functional sentences comes from making certain properties manifest.[7]

The completed truth table of a sentence wears many of the sentence's truth-functional properties on its sleeves. These include whether the sentence is a tautology, a contradiction, or contingent (i.e. true under some but not all truth-value assignments). For instance, to determine if a sentence is a tautology, it suffices to check whether it is true under every possible truth-value assignment to its atomic sentence letters. This epistemic dependence relation supplies an appropriate plan for determining whether a sentence is a tautology. The completed truth table makes manifest the sentence's truth-values, e.g. by collecting them under the sentence's main connective. In other words, given the completed truth table, one ought to infer the sentence's truth-value for every truth-value assignment to its atomic sentence letters. Then, by applying the preceding EDR for a tautology, the truth table makes manifest whether the sentence is a tautology. The truth table thereby wears this property on the sleeves, namely, the property of being a truth table of a tautology. Provided that one sees the truth table and applies this EDR, they ought to infer that the sentence is a tautology.

Some sentences wear their tautological status on their sleeves all by themselves, no truth table needed! As a simple example, consider the sentence $(p \vee \neg p) \wedge (\neg r \vee q \vee s \vee \neg q)$, which is in conjunctive normal form (CNF). To see that this sentence is a tautology, we

[6] Using an ellision introduced below, we could elide "sleeve properties" to the simpler "manifest properties," where this notion would now encompass properties manifest to the 0th or 1st degree.

[7] The following discussion draws heavily upon Goldfarb (2003, pp. 67, 73–74), which first exposed me to the idea of an expression wearing a property on its sleeves.

can rely solely on features of it that are already manifest. For instance, it is manifest that the sentence consists of two conjuncts, each of which is a disjunction of negated and unnegated sentence letters. Anyone who understands the sentence ought to infer these surface-level properties; they are trivially manifest—what we might call 'manifest to the 0th degree.' Moreover, any agent who knows the following EDR also ought to infer that the sentence is a tautology: to determine if a conjunction of disjunctions is a tautology, it suffices to check whether each conjunct contains a sentence letter that occurs both negated and unnegated. In the first conjunct, it is manifest that $p$ occurs negated and unnegated, whereas $q$ occurs negated and unnegated in the second conjunct. Hence, each conjunct is a tautology, so the sentence itself is a tautology. Combined, the sentence and this EDR make manifest that the sentence is a tautology (we might say that this property is 'manifest to the 1st degree'). In general, any sentence in conjunctive normal form wears the property of being a tautology (or not) on its sleeves. We simply use the following epistemic dependence relation: to determine whether a sentence in CNF is a tautology, it suffices to check whether each conjunct contains a sentence letter and its negation.

A sentence is in disjunctive normal form (DNF) provided that it is a disjunction of conjunctions of sentence letters or their negations, such as the following sentence: $(p \wedge q) \vee (\neg s \wedge r) \vee (\neg p \wedge p)$. DNF makes manifest whether a sentence is *satisfiable*, i.e. is true on some truth-value assignment. This follows from logical properties of disjunctions and conjunctions. A disjunction is satisfiable if and only if it has a satisfiable disjunct. In DNF, each disjunct is a conjunction. Hence, we note further that a conjunction is satisfiable if and only if it is not truth-functionally equivalent to a contradiction, such as "$p \wedge \neg p$." Collectively, these two facts yield the following epistemic dependence relation: to determine whether a sentence in DNF is satisfiable, it suffices to check whether at least one disjunct does not contain a sentence letter and its negation. Similarly, to determine whether a sentence in DNF is unsatisfiable (i.e. false on every truth-value assignment), it suffices to check whether every disjunct contains a sentence letter and its negation (in which case every disjunct is a contradiction). Provided we implement these two EDRs, a sentence in DNF wears its satisfiability or unsatisfiability on its sleeves. More generally, disjunctive normal form makes manifest the truth-value assignments on which the sentence is true (it wears these assignments on its sleeves).

**Manifest Lorentz Covariance**

Physicists commonly refer to some expressions as being "manifestly Lorentz covariant." For instance, the following equation is manifestly Lorentz covariant: $a_\rho a_\nu b^{\rho\mu} = B^\mu_\nu$. This simply means, I will argue, that this expression wears the property of Lorentz covariance on its sleeves. In conjunction with an appropriate EDR, one ought to infer that this expression is Lorentz covariant, *solely on the basis of properties that are already manifest.*

A suitable plan for checking whether an expression is Lorentz covariant comes from the following fact: an equation in tensor form is Lorentz covariant provided that i) non-repeated upper and lower indices on either side match and ii) repeated indices appear once lower and once upper on the same side of the equation. This fact yields the following EDR: to check whether a tensor equation is Lorentz covariant, it suffices to check whether these two conditions are met. Notice that these conditions rely on properties of the equation that are already manifest, namely the occurrence and placement of indices. Hence, an agent who understands this EDR and sees the expression ought to infer that the equation is Lorentz covariant. Likewise for any other expression that satisfies these conditions. Such expressions wear Lorentz covariance on their sleeves. (For a non-conscious agent, we can replace talk of 'seeing' and 'understanding,' with notions of being given the expression as *input* and *implementing* this problem-solving plan.)

My account also illuminates what it means to say that an expression is manifestly Lorentz invariant. When physicists say this, they simply mean that the expression wears Lorentz invariance on its sleeves. An example is the expression $F_{\mu\nu}F^{\mu\nu}$. Here, each lower index is paired with a matching upper index, and there are no free indices. These manifest facts suffice for inferring that the expression transforms as a scalar under Lorentz transformations. Hence, in conjunction with this problem-solving plan, the expression $F_{\mu\nu}F^{\mu\nu}$ wears its Lorentz invariance on its sleeves.

In contrast, some expressions are Lorentz invariant, but this property is not worn on the sleeves (it is hidden). One can prove that such expressions transform as a scalar, but it is not the case that one ought to infer this *solely on the basis of properties that are already manifest.* Instead, one must rely on properties that become manifest only after starting the proof. A well-known example is the Lorentz invariant measure $\int \frac{d^3k}{(2\pi)^3 2w_k}$, where $w_k = +\sqrt{|k|^2 + m^2}$. One can prove that this measure is invariant under proper orthochronous Lorentz transformations. By the end of this proof, its Lorentz invariance is manifest. But the expression itself does not wear this property on its sleeves, in the way that "$F_{\mu\nu}F^{\mu\nu}$" does. At least, I do not know of any appropriate EDR that makes this property manifest solely on the basis of properties of "$\int \frac{d^3k}{(2\pi)^3 2w_k}$" that are already manifest.

These examples illustrate a general epistemic difference between expressions that wear a property on the sleeves vs. those that do not (but that still possess the property). In both cases, to make the property manifest, we must engage in problem-solving. We must apply an epistemic dependence relation(s) that forms the basis of a problem-solving plan. When the property is worn on the sleeves, we do not need to consider any intermediary expressions. The expression itself contains sufficient information for determining whether the property obtains. In contrast, when the property is not worn on the sleeves, we must construct intermediary expressions, such as a truth table. It is from these intermediaries that the property ultimately becomes manifest (i.e. at the end of problem-solving). Section 5.4 analyzes this kind of epistemic difference in terms of a difference in the ruling out of epistemic possibilities. A formulation that makes a property manifest rules out possibilities that the non-manifest formulation does not. This kind of epistemic difference contributes to the non-practical epistemic value of making properties manifest, i.e. to its intellectual significance.

From these considerations, we begin to see how one could construct a gradated notion of manifest properties. Clearly, there is a sense in which properties that are worn on the sleeves are *more manifest* than those that are not. Moreover, we have seen that some properties are trivially manifest, and thereby trivially worn on the sleeves. Above, I referred to these as being "manifest to the 0th degree." Properties that are non-trivially worn on the sleeves are "manifest to the 1st degree": we make sleeve properties manifest by relying on an EDR that exploits only properties that are 0th-degree manifest. Often, when scientists and mathematicians talk about "manifest properties," they really mean properties that are non-trivially worn on the sleeves. Such properties are not immediately manifest, but they become manifest once we apply an EDR that relies solely on already manifest properties. Section 5.4.3 develops a more general proposal for a gradated account of manifest properties.

## 5.3 Manifest Meanings

As discussed briefly in Section 1.6, languages can differ in how manifest they make the meaning of a word. At first glance, the German word "die Speisekarte" is completely synonymous with the English word "the menu." Both mean what we can denote at the level of thought by 'MENU.' Yet, due to the sub-word structure of "die Speisekarte," German makes the meaning of this word more manifest than English. On my account, this means that there are problem-solving contexts where a German speaker ought to infer the meaning of "die Speisekarte," whereas an English speaker in the same (non-linguistic) epistemic circumstance ought not infer the meaning of "menu."

Consider two agents, Gertrude and Ender, who are native speakers of German and English, respectively. Gertrude has forgotten the meaning of "die Speisekarte" while Ender has forgotten the meaning of "menu." Thanks to the semantic substructure of "die Speisekarte," Gertrude is in an epistemically superior position. In German, "die Speise" means DISH or FOOD, while "die Karte" means CARD or CHART. Hence, Gertrude ought to increase her credence that "die Speisekarte" means a card or chart that displays dishes or food, i.e. that it means MENU. In contrast, Ender is not permitted to make a similar inference. Knowing the meanings of "dish" and "card" is of no use here, since the English word "menu" does not have an analogous substructure. On the basis of what he can remember, Ender ought not increase his credence that "menu" means MENU. Hence, German supports a problem-solving plan that English does not. In virtue of this plan, German makes manifest the meaning of "die Speisekarte," whereas English does not make manifest the meaning of "menu."[8]

In order for Ender to carry out Gertrude's problem-solving plan, Ender would effectively need to 'change variables' by translating into German. For instance, Ender would need to know that the English word "menu" is synonymous with the German "die Speisekarte," and that "die Speise" means DISH/FOOD while "die Karte" means CARD/CHART. Given this additional information, Ender ought to increase his credence that "menu" means MENU. But notice how Ender requires knowledge of a translation procedure, whereas Gertrude does not. This provides another way of seeing that German,

[8]Similar examples of this kind include the German for 'attractions'—die Sehenswürdigkeiten (things worthy of seeing)—and for 'deranged'—geistesgestört (distortion of the mind or spirit). The German for 'attractions' makes manifest that these are things that are typically worth seeing.

but not English, makes the meaning of this word manifest.

Of course, there are other ways to make the meaning of a word manifest. For anyone with sufficient background knowledge, a dictionary makes manifest the meaning of unknown words. Ender could look up the meaning of "menu" in an English dictionary. Its meaning would be made manifest by a definition such as this: "a list from which to request food dishes at a restaurant or social event." Provided that Ender knows the meanings of enough of these words, he ought to increase his credence that "menu" means MENU. Gertrude could likewise follow this alternative problem-solving plan, consulting a German dictionary for the meaning of "die Speisekarte."

Perhaps one might worry at this point: is there really any philosophically interesting difference between Gertrude inferring the meaning of "die Speisekarte" from the meanings of its sub-words vs. inferring its meaning from a dictionary? Indeed, in both cases, Gertrude infers the meaning of "die Speisekarte" on the basis of knowing other words. As we have seen above in the context of sleeve properties, there is at least one important difference. Through the former problem-solving plan, the word "die Speisekarte" makes its meaning manifest, solely using features of it that are already manifest (namely, its sub-word structure). A German does not need a German dictionary for this. Whereas in the latter problem-solving plan, a dictionary does the work (indeed, a good dictionary does this work in any language, for any word—at least for speakers with sufficient knowledge of the language). Relative to the problem-solving plan that relies on a dictionary, there is no epistemic difference between Gertrude and Ender. Relative to the problem-solving plan that involves a decomposition into sub-words, an epistemic difference arises.

This example illustrates that it is not the expressive means on its own that makes a property manifest. Rather, it is the expressive means in conjunction with a problem-solving plan. Whether a property is made manifest depends on how one plans to use an expressive means. Ender could make the meaning of "menu" manifest if he chooses to use an English dictionary. But Gertrude does not need a dictionary, provided that she plans to infer the word's meaning from known sub-words. This example is particularly striking because it shows how the problem-solving plans that are available can depend on the choice of expressive means, e.g. language or notation. Ender is not even able to carry out the sub-word decomposition plan that Gertrude follows. If Ender tries to apply this EDR, it takes him nowhere, for "menu" does not decompose into English sub-words.

## 5.4 The Value of Making it Manifest

Having expounded my account of what it means to make a property manifest, I return now to this chapter's central question: what is the value of making properties manifest? More precisely, what is required for it to be valuable? This question is a special case of Chapter 1's investigation into the value of compatible reformulations. As before, at least three dimensions of value suggest themselves: instrumental/practical, metaphysical, and non-practical epistemic (what I am calling 'intellectual' value). After presenting instrumentalist and fundamentalist accounts of the value of manifest properties, I propose a conceptualist middle ground.

Both instrumentalism and fundamentalism give straightforward criteria for when it is valuable to make a property manifest. According to instrumentalism, making a property manifest is valuable whenever it contributes to the achievement of other scientific aims. For instance, provided that making a property manifest makes problem-solving more convenient or efficient, it is valuable by the lights of instrumentalism. The instrumentalist denies that making a property manifest ever constitutes *on its own* the achievement of a scientific aim. The most austere form of instrumentalism—conventionalism—contends that making properties manifest is *merely convenient*.

In contrast, fundamentalism contends that making a property manifest can constitute the realization of an aim of science, namely the aim of describing reality in ever more fundamental terms. On this view, a variable choice is valuable at least when it leads to a more fundamental or joint-carving description of a given phenomenon. Consequently, making a property manifest is valuable whenever doing so constitutes a more fundamental description of the phenomena. *Ceteris paribus*, a variable choice that makes a more fundamental property manifest qualifies as more valuable than a choice that obscures such a property.

To fare at least as well as these accounts, conceptualism must provide clear criteria for when it is valuable to make a property manifest. Such criteria must underwrite an evaluative asymmetry between those variable choices that make a property manifest vs. those that do not. The former are better or more valuable than the latter, other things equal. Fortunately, an empiricist-friendly, epistemic criterion lies ready at hand, which I articulate in Section 5.4.1.

As a warm-up, consider the simple case where our epistemic end is to determine whether an expression or system possesses a particular property. By making that property manifest, we achieve our epistemic end. In this context, making a property manifest constitutes the achievement of our goal. Insofar as achieving this goal is epistemically valuable, so is making the property manifest. Variable choices that fail to make the property manifest fall short of this goal, and are to that extent less valuable. Such variable choices do not preclude us from obtaining this knowledge, but they do not suffice for it.

To see this, recall the graph theory example from Section 5.2. By making planarity manifest, we already achieve our aim of determining whether the graph is planar. In virtue of this property being manifest, we ought to infer planarity. In contrast, a non-planar representation of the graph does not suffice for achieving our aim. Doing so requires a further epistemic transformation, such as constructing a planar representation from the non-planar one.

My conceptualist account has important differences with both instrumentalism and fundamentalism. In contrast with instrumentalism, making a property manifest is not *merely* an instrument for achieving scientific aims. Instead, it can constitute the achievement of epistemic aims, such as knowing whether or not a system has a particular property.

Additionally, the kind of epistemic value that conceptualism identifies is logically independent from what is all-things-considered most practically valuable. In the context of determining the meaning of 'menu,' it will typically be more convenient to use an English dictionary than to translate 'menu' into German, learn the meanings of some German subwords, and then translate back. Nevertheless, there remains a sense in which German is epistemically better suited to solve this problem. Similarly, we can imagine contexts where someone with perfect pitch would prefer to use a measurement device to determine the pitch of a sound. Perhaps the sound is extremely loud, and they desire to protect their ears by measuring the sound while in a different room. Hence, the conceptualist criterion for significance has nothing intrinsically to do with speed, convenience, or other practical dimensions of value.

The 20th century Russian physicist Vladimir Fock's commitment to Marxism provides an illuminating historical example. Motivated by dialectical materialism, Fock developed harmonic coordinates as a preferred coordinate system for expressing equations in general relativity (Graham 2000, p. 34). One can imagine it being prudent—in certain political contexts—to prefer Fock's formulation regardless of its epistemic benefits. Vice versa, one might sometimes prefer to use harmonic coordinates for their intellectual advantages, even while denouncing Marxism in all its forms.[9]

In contrast with fundamentalism, conceptualism contends that the value of making a property manifest does not depend on that property being relatively fundamental. In the simple case of checking whether a system has a property, all that matters is that we have a prior epistemic aim of determining whether the system has this property. The conceptualist account applies to any kind of property of interest, regardless of whether such properties are relatively fundamental. The same sorts of epistemic differences can arise for properties that are completely non-fundamental or 'gruified.'

This flexibility presents one of the chief advantages of conceptualism over fundamentalism. At least part of the epistemic value of making properties manifest floats free from the relative fundamentality of those properties. In many contexts, none of the properties that we make manifest seems to be most metaphysically fundamental. Section 5.5 provides a simple example stemming from the choice of Cartesian vs. polar coordinates. It is implausible that one set of coordinates counts as 'metaphysically more fundamental' than another. After all, coordinates are ways of representing states of affairs, rather than properties of those states of affairs. Nonetheless, the kinds of epistemic differences that arise in these cases completely parallel the differences that arise in cases where fundamentalism might get traction, such as the case of hidden symmetries or gauge choices. Yet as Section 5.7 shows in the context of gauge choices in quantum field theory, making one (fundamental) property manifest often comes at the expense of obscuring others. To assess the relative value of these gauge choices, the fundamentalist requires some way of comparing these trade-offs. Conceptualism does not require this kind of accounting in order to make sense of why it can be valuable to make different properties manifest in different contexts.

### 5.4.1 Ruling out epistemically possible solutions

Of course, in many cases our task is more complicated than simply checking whether or not an expression has a property. Section 5.5 illustrates one such context, where the task

[9]Thanks to Gordon Belot for suggesting this example.

is to determine the equation of a line. For a horizontal line, I argue that it is intellectually valuable to make the vertical degree of freedom manifest, although this is not the same as determining the equation of the line. Hence, we need a more general criterion for the intellectual value of making properties manifest. For ease of discussion, I introduce some terminology: let's call a formulation or choice of variables that makes a property (more) manifest a "(more) manifest formulation." Conversely, let's call a formulation that fails to make a property manifest a "non-manifest formulation" (or at least a "less manifest" one).

In general, a manifest formulation has the following epistemic advantage: *it rules out epistemically possible solutions that a non-manifest formulation does not.* To see this, it helps to reconsider some prior examples. A planar representation of a planar graph rules out the epistemic possibility that the graph is not planar. In contrast, when presented with a non-planar representation, it remains epistemically possible that the graph is not planar. Consider next a person with perfect pitch. In virtue of pitch being manifest to them, they immediately rule out epistemic possibilities that an ordinary person can rule out only via a measurement device. Clearly, there is an epistemic advantage to having perfect pitch, even for someone who chooses not to use it for practical reasons. Similarly, when we present an expression in manifestly covariant form, we rule out the epistemic possibility that the expression is not Lorentz covariant. A non-covariant form leaves open this epistemic possibility, in the sense that it is not the case that we ought to infer that the expression is Lorentz covariant.

Of course, what matters isn't simply the *number* of epistemic possibilities that are ruled out. What matters is ruling out epistemically possible *solutions*, rather than epistemic possibilities *tout court*. Relative to the aim of solving a particular problem, there is little-to-no value in ruling out epistemic possibilities that have nothing to do with that problem. If I am trying to determine whether a sentence is a tautology, I do not advance by noting that I am wearing pink socks (despite the fact that this observation rules out many epistemic possibilities). Indeed, as the examples in Sections 5.5 and 5.7 demonstrate (concerning coordinate choices and gauge choices, respectively), a formulation that is non-manifest with respect to one property can be manifest with respect to another. Hence, if we were to naïvely count the epistemic possibilities that such formulations rule out *tout court*, we would miss key epistemic differences between them.

I am proposing that the value of making a property manifest derives from ruling out possibilities that we could rationally entertain in the course of problem-solving.[10] Unsurprisingly, the class of possibilities that matters changes across different kinds of problems. As Section 5.4.3 discusses, a problem's epistemically possible solutions constitute a 'search space' for that problem. Making a property manifest is epistemically valuable when it constrains this search space.

As noted above, conceptualism does not deny the instrumental or practical value of making properties manifest. Indeed, this practical value often stems from the conceptualist criterion I have just proposed: by ruling out more epistemically possible solutions, manifest formulations are often more convenient for problem-solving. By eliminating these possibilities, it typically becomes easier or faster to solve a problem. Of course, other practical considerations can intervene, such as the pedagogical costs of learning a manifest formulation. Hence, as I have been arguing, these are genuinely independent dimensions of value. Although greater convenience is often a symptom of intellectual significance, it is not a criterion.

Section 5.9 provides a striking illustration of this moral, involving a kind of case that arises frequently with symmetries. By reformulating such that a symmetry is put on the sleeves, we gain the ability to construct increasingly complicated expressions that manifestly respect this symmetry. In this case, we gain the ability to build more complex invariants out of starting points that are manifestly invariant under the symmetry. Unsurprisingly, this ability is incredibly convenient in many contexts. It is so convenient that it is easy to lose sight of its underlying intellectual significance, which obtains independently of these practical benefits. By making the symmetry into a sleeve property, one ought to infer that a given expression has that symmetry. Whereas otherwise, it would be a live possibility that the expression is not invariant. Due to this epistemic possibility, it would be necessary to check—via calculation—that the expression has the symmetry in question. Section 5.9's example, involving supersymmetry, also makes salient the fact that making a symmetry into a sleeve property can take *a lot* of work. In many contexts, it would not be practically worth doing this work, unless one was faced with multiple problems that could practically benefit from it.

[10]Such rationally-entertainable possibilities just are the epistemically possible solutions. If we follow Gibbard (1990) in giving an expressivist treatment of rationality, then judgments about possible solutions are inherently normative.

### 5.4.2 Less surprising, more intelligible

By ruling out epistemic possibilities, a more manifest formulation makes the phenomena of interest less surprising or mysterious. If we start with a more restricted space of possible solutions, the fact that the solution has a given property is typically less surprising than it otherwise would be. (If we were to apply a principle of indifference, we would begin with different priors concerning the property of interest, depending on whether we start within a more manifest formulation vs. a less manifest one.) I take this decrease in surprise to be a sufficient condition for greater intelligibility. More manifest formulations often make the solution to a problem more intelligible, at least by typically decreasing surprise. Assuming that science aims to make phenomena as intelligible as possible, we gain a non-practical epistemic reason to prefer formulations that make a given phenomenon more manifest.

A simple example from quantum field theory illustrates these connections between epistemic possibilities, surprise, and intelligibility. The Lagrangian density below initially appears to describe an interacting scalar field $\phi$, due to the terms third-order and higher, such as $\phi^3$ (Cheung 2017, p. 2):

$$\mathcal{L} = \frac{1}{2}[1 + \lambda_1\phi + \frac{1}{2!}\lambda_2\phi^2 + \frac{1}{3!}\lambda_3\phi^3 + \ldots]\partial_\mu\phi\partial^\mu\phi \tag{5.4.1}$$

Written in this form, the Lagrangian density leaves open the possibility that it describes an interacting field. It is not the case that one ought to know whether the amplitudes that describe scattering $n$-many particles vanish. We might then go on to calculate the 'tree-level' amplitude for scattering four particles (i.e. to first order in perturbation theory). We would find that it vanishes, reflected by the cancellation of a few Feynman diagrams. Intrigued, we might press on, calculating 5-point amplitudes and higher. We would find that each vanishes. As Cheung notes, "the 14-particle amplitude also vanishes, albeit through the diabolical cancellation of upwards of 5 trillion Feynman diagrams" (2017, p. 3). Well before this point, we might already suspect that the Lagrangian density (5.4.1) actually describes a free scalar field.

Indeed, by performing a suitable field redefinition, we can transform the density (5.4.1) into one that manifestly describes a free scalar field.[11] A manifestly free theory rules

[11] In general, scattering matrix elements are invariant under field redefinitions $\phi \to f(\phi)$ such that f'(0) =1 (Cheung 2017, p. 3). For at least this reason, we ought not take the Lagrangian density or the Feynman rules for vertices too literally!

out the possibility that there are non-vanishing amplitudes describing particle scattering. Hence, the vanishing of these $n$-point amplitudes becomes unsurprising and to that extent more intelligible. We expect that a free scalar has trivial interactions with itself. In a claim consilient with many themes of this chapter, Cheung notes that "a poor choice of field basis may obscure or *altogether conceal* certain underlying structures of the theory" (2017, p. 3).

A similar moral arises in the context of conjunctive normal form and tautological sentences. Given a structurally complicated or 'concealed' tautology, we might check whether it is a tautology by computing each row of its truth table. As we proceed, we might begin to suspect that we are dealing with a tautology. The truth-value of certain rows might initially seem surprising. By contrast, if we were to convert this sentence into a logically equivalent conjunctive normal form, then its tautological status would be manifest. It would then be unsurprising that each row of its truth table evaluates to true. The possibility of any row evaluating to false would have already been ruled out.

### 5.4.3 Degrees of manifestness

The connection between i) making a property manifest and ii) ruling out epistemic possibilities suggests a promising strategy for gradating the notion of manifestness. A formulation makes a property manifest to the extent that it rules out possibilities where the property does not obtain. For instance, consider a musician who has 'good but not perfect pitch,' someone who can typically identify a tone to 'plus or minus' the actual pitch-class. Intuitively, the pitch is more manifest to them than to someone who completely lacks a musical ear. On the criterion I am proposing, this is because a musician with good-but-not-perfect pitch rules out more epistemically possible solutions than an ordinary person.

To make this criterion precise, we require a measure on the space of epistemically possible solutions. To determine which of two formulations or variable choices makes a given property more manifest, we must compare the possible solutions that they rule out. Perhaps there is a uniform way of quantifying such epistemic possibilities.[12] Regardless, it seems that we can at least suggest plausible measures in many problem-solving contexts. For instance, when it comes to determining whether a truth-functional sentence

[12] Perhaps such criteria could be constructed from information theory, topology, or truth-maker semantics.

is a tautology, each row of the truth table contributes two epistemic possibilities: true or false under that truth value assignment. A choice of expressive means that rules out more of these possibilities counts as making a given property more manifest.

Moreover, there seem to be good independent reasons for taking seriously the idea of "a space of epistemically possible solutions" for a problem. Generically, we can understand problem-solving as a process of structuring a space of possible solutions. Epistemically-different problem-solving plans result in different structurings of this space: they rule out or in different possibilities. Other things equal, we have epistemic reasons to prefer those problem-solving plans that restrict the space of solutions as much as possible. Applying the account of better understanding from Section 3.5, this means that manifest formulations provide better understanding of the phenomena (i.e. we have a non-practical epistemic reason for preferring a manifest formulation). The same kind of reasoning applies to the use of symmetry groups of differential equations: identifying such groups epistemically constrains the solutions of differential equations that obey those symmetries. This is one of the insights that led Wigner to apply symmetries to quantum mechanics in the 1920s.

At least in some scientific contexts, the space of possible solutions seems highly concrete and far from metaphorical. Physicists provide precise characterizations of such epistemic possibilities whenever they construct a space of possible values for an unknown parameter. Many experimental searches in cosmology and particle physics aim to restrict this space of epistemically possible values as much as possible. Although we may not be able to achieve this level of precision in an arbitrary problem-solving context, it at least supplies a helpful model for philosophical theorizing about reformulations.

### 5.4.4 Problem-solving adequacy and fruitfulness

In many of the examples from Sections 5.2 and 5.3, the more manifest formulation makes available a problem-solving plan that a less manifest formulation does not support. This is particularly striking in the case of 'sleeve properties.' By wearing a property on the sleeves, the manifest formulation allows us to solve the problem by 'reading off' this property from the expression. For instance, a manifestly Lorentz covariant expression supports a problem-solving plan for Lorentz covariance that a non-manifestly covariant expression does not support (at least without further transformations). Similarly for the case of German vs. English: German makes available a problem-solving plan for guessing the meaning of ‘die Speisekarte’ that English does not support for ‘menu.’

By making alternative problem-solving plans available, these kinds of reformulations contribute to the aim of problem-solving adequacy (introduced in Chapter 4). They supply plans that can succeed in a wider variety of epistemic circumstances. For instance, when it comes to solving the ‘menu’ problem, a German speaker does not need a dictionary. Likewise, when it comes to determining the pitch-class of a tone, someone with perfect pitch does not require a measuring device.

I conjecture that a more manifest formulation supports alternative problem-solving plans in virtue of ruling out more epistemically possible solutions. Because the English language does not place constraints on the meaning of ‘menu’ from English subwords, a pure-English speaker has no other recourse than to consult a dictionary (or some other testimonial source). In contrast, German lets us decrease credence in many epistemic possibilities for the meaning of ‘die Speisekarte,’ such that no dictionary is necessary (at least not necessary for increasing our credence in the meaning of this word).

These differences in problem-solving adequacy amount to differences in fruitfulness. The more manifest formulation supports a plan that can succeed in a wider range of problem-solving contexts, such as contexts where we lack a measuring device for tone or lack a dictionary. Recall that in Chapter 1, I argued that bald appeals to fruitfulness do not provide a satisfying account of the intellectual differences between reformulations. Instead, I urged seeking a local understanding of these differences. Wherever possible, we ought to be able to appraise compatible formulations within a shared domain of problem-solving. Here, we see a local strategy for accounting for differences in fruitfulness: at least some such differences seem to arise from differences in the ruling out of epistemically possible solutions.

## 5.5 Coordinate Transformations

Coordinate transformations provide one of the simplest cases of philosophically interesting variable changes. Different kinds of coordinate systems sometimes make different properties manifest. Below, I will demonstrate why this matters, using two-dimensional Cartesian vs. polar coordinates as a detailed example. We will see that Cartesian coordinates make manifest the properties of being horizontal or vertical, whereas polar coordinates make manifest the properties of having constant polar angle or constant radius (i.e. being a circle). Other examples of intellectually significant coordinate choices include rectangular vs. spherical vs. cylindrical coordinates in three dimensions, Cartesian vs. internal coordinates in the modeling of molecules, and Eulerian vs. Lagrangian coordinates in fluid dynamics. Although we can express many of the same epistemic dependence relations in these coordinate systems, different coordinate choices nevertheless lead to differences in what we need to know to solve problems.

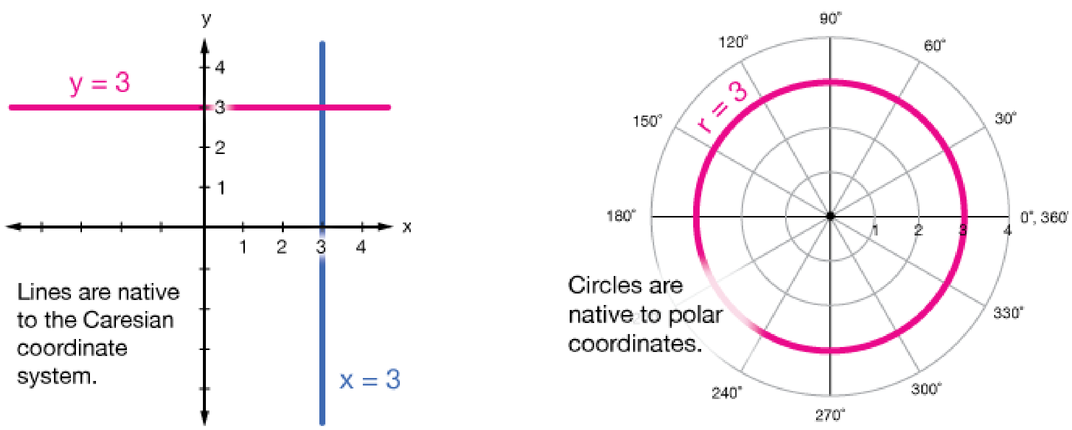
Of course, not all coordinate transformations are intellectually significant. Some coordinate transformations are instead trivial notational variants: they may provide differences in convenience (up to our idiosyncratic conventional preferences), but they evince no intellectually significant differences. As described in Section 1.6, transforming between two Cartesian coordinate systems typically does not provide any differences in EDRs. Such transformations are analogous to systematically replacing every instance of the numeral "5" with "V" in our numeral system. This kind of notational change does not alter what we need to know to solve problems.

One case where coordinate transformations *do* seem to make an intellectual difference is when a system has a symmetry or invariant. For instance, if we are modeling a cylinder, then it is intellectually significant to passively transform to cylindrical coordinates where the z-direction lies along the length direction of the cylinder (so that circular cross-sections of the cylinder are perpendicular to this axis). This makes manifest the length of the cylinder. More precisely, the z-axis now wears the cylinder's length on its sleeves. Such choices amount to a separation of degrees of freedom. Indeed, the examples below involving Cartesian vs. polar coordinates illustrate the same moral. Cartesian and polar coordinates are adapted to equations with different kinds of symmetries or invariant degrees of freedom.

### Cartesian vs. Polar Coordinates

Although there is a simple translation procedure between Cartesian and polar coordinates, these coordinate systems are not trivial notational variants. For certain problems, these notations support epistemically different problem-solving procedures. In virtue of these differences in problem-solving plan, Cartesian and polar coordinates make different properties manifest. According to my account of manifest properties, this means that they change when we ought to infer that a system has a given property.

Figure 7 illustrates the different properties that Cartesian and polar coordinates make manifest.[13] These properties define different kinds of graphs. Given a horizontal or vertical line, Cartesian coordinates make manifest the relevant invariant degrees of freedom (the y-coordinate and x-coordinate, respectively). Likewise, given a circle or a diagonal line, polar coordinates make manifest the relevant invariant degrees of freedom (the radius and polar angle, respectively).




(a) Cartesian coordinates make manifest horizontal and vertical lines.

(b) Polar coordinates make manifest circles and diagonal lines, e.g. those through the origin.

Figure 7: Cartesian vs. Polar Coordinates

By *Cartesian coordinates*, I mean the following expressive means: a choice of $x$ and $y$ axes has been made on the plane, with a right angle between them (so that the axes are orthogonal). To represent the equation of a line, we must represent it using these variables $x$ and $y$. We can imagine working on standard grid paper, representing many distance measurements that we can make using a ruler. By *polar coordinates*, I mean the following expressive means: a choice of reference axis has been made from which to measure the polar angle $\theta$. A choice of origin has been made from which to measure the radial distance $r$. Again, we can imagine working on polar grid paper, with circles of increasing radii surrounding the origin, and various polar angles indicated with diagonal lines passing through the origin. In both cases, to describe multiple functions at once in a commensurable way, we must keep the reference choices fixed. Hence, in solving the

[13] This image comes from `https://xaktly.com/MathPolarCoordinates.html`.

problems below, it is impermissible to alter the reference choices (e.g. placement of the $x$-axis, $y$-axis, angular reference axis, or the origin). This prevents us from trivializing a given problem simply by making a convenient choice of reference axis.

To identify epistemic differences between Cartesian and polar coordinates, I will compare two agents: Carla and Paula. Carla works within a Cartesian coordinate system, whereas Paula works within a polar coordinate system. They are engaged in solving various problems in Euclidean geometry. The philosophical challenge is to locate differences in what these agents need to know at various stages of problem-solving—differences that go beyond the stipulated fact that Carla understands Cartesian coordinates, while Paula understands polar coordinates. If there were no such differences, then Cartesian and polar coordinates would be trivial notational variants after all, in the same way that the English "here is a dog" is synonymous with the German "hier ist ein Hund". Of course, there is trivially an epistemic difference between knowing English and knowing German, but as Section 1.6 describes, that kind of language-dependent epistemic difference does not qualify as intellectually significant.

Here is the first problem: you are presented with a horizontal line drawn in your coordinate system. What is the equation of this line? There are a variety of different ways to proceed, based on different epistemic dependence relations. Using point–slope form, it suffices to know two points on the line, subsequently using these to calculate the slope and intercept of an axis. Alternatively, since the line is horizontal, it suffices to express its vertical displacement from a reference line. Imagine that both Carla and Paula plan to rely on this latter EDR. The question then is whether in executing their plans, any differences arise in which facts are manifest. Specifically, is there a point at which Carla, but not Paula, ought to infer the equation of the horizontal line?

Suppose that Carla and Paula begin in the same way, measuring the vertical displacement of the horizontal line using a ruler (or, perhaps Carla uses the markings on her $y$-axis, Paula the markings on her $r$-axis). It turns out that for the given line, the vertical displacement is 5 units from their respective reference lines. At this point, I contend, their problem-solving plans diverge. Since the vertical displacement just is her $y$-coordinate, Carla ought to infer that the equation of the line is $y = 5$, thereby arriving at the Cartesian solution. In contrast, Paula cannot yet express the equation of the line in her coordinate system, despite knowing that the vertical displacement is 5 units. Paula needs to know something further, namely she needs to know how to express vertical displacement in polar coordinates. Specifically, Paula needs to know that $y = rsin\theta$, relating the height of a right triangle to its hypotenuse and the angle opposite the height. In contrast, Carla did not need to invoke any translation procedure. There is thus a difference in what Carla and Paula need to know, even once we control for language-dependent epistemic differences. Compare the structurally parallel example from Section 5.3: Ender would need to translate 'menu' into German in order to carry out Gertrude's problem-solving plan (which relies on the linguistic substructure of 'die Speisekarte').

This example shows that Cartesian coordinates make manifest the property of being horizontal. A line is horizontal whenever its vertical displacement is invariant. Cartesian coordinates focus attention on the vertical displacement as one of the basic degrees of freedom, namely the coordinate $y$. They trivially wear vertical displacement on the sleeves. Hence, upon measuring the vertical displacement of a horizontal line, Carla ought to infer the equation of this line in Cartesian coordinates. Since polar coordinates do not focus on the vertical displacement as one of the basic degrees of freedom, it is not the case that Paula ought to infer the equation of the line in polar coordinates. Indeed, it is tempting to make the stronger claim that it would be impermissible for Paula to infer the equation of the line in polar coordinates until she performs this translation. Arguably, Paula needs to know how to express the vertical displacement in polar coordinates. This effectively involves translating from the Cartesian coordinate $y$ to polar coordinates. *Mutatis mutandis*, we see that Cartesian coordinates also make manifest the property of being vertical, i.e. of having invariant horizontal displacement from a reference line.

Polar coordinates make different properties manifest.[14] These include the properties of i) having constant polar angle and ii) having constant radius (being a circle). Imagine that Carla and Paula are presented with a diagonal line passing through the origins of their respective coordinate systems. Both plan to exploit the following epistemic dependence relation: to determine the equation of a diagonal line through the origin, it suffices to measure the angle between it and a given reference line (the $x$-axis in the case of Carla; the $\theta = 0$ axis in the case of Paula). To keep things as epistemically symmetric as possible, suppose that both use a protractor to measure the angle, determining that it is 45 degrees.

[14]Such properties are not always mutually exclusive. Both Cartesian and polar coordinates make manifest the defining property of a vertical line through the origin: it has both zero horizontal displacement and constant polar angle 90 degrees.

At this point, Paula ought to infer that the equation of the line is $\theta = 45°$. The equation of the line is already manifest to her.

In contrast, Carla is not yet permitted to infer the equation of the line in Cartesian coordinates (namely, the fact that $y = x$). Instead, she needs to know a further fact, namely how to relate this reference angle of 45 degrees to an expression involving the Cartesian coordinates $x$ and $y$. Carla has effectively measured the polar angle $\theta$, and she needs to know how to translate this angular degree of freedom into Cartesian coordinates. Specifically, she needs to know that $\theta = arctan(y/x)$. From this equation, she can infer that $y/x = tan(\theta) = tan(45°) = 1$. After this series of inferences, Carla ought to infer that $y = x$, thereby solving the problem in Cartesian coordinates. Although Carla exploited the same initial plan as Paula—namely, the directive to measure the angle that the line makes with a reference line passing through the origin—she required additional knowledge to solve the problem, knowledge that Paula did not require in polar coordinates.

*Mutatis mutandis*, the same lesson applies to circles centered at the origin. Since these geometric objects have constant radii, polar coordinates make their equations manifest. For instance, upon measuring the radius of such a circle to be 5 units, Paula ought to immediately infer that its equation is $r = 5$. In contrast, Carla needs to know how to relate this radius to Cartesian coordinates, using the trigonometric fact that $r = \sqrt{x^2 + y^2}$.

These examples evince subtle epistemic differences in the choice of expressive means. To appreciate them, it may help to recall the case of a person with perfect pitch. When it comes to horizontal and vertical lines, Carla is like someone with 'perfect pitch' for these. Upon a minimal measurement (analogous to hearing the pitch), she ought to immediately infer the equation of the line. Likewise, Paula has 'perfect pitch' for diagonal lines through the origin and circles centered at the origin. Upon measuring the polar angle or radius, Paula ought to immediately infer the equations for these kinds of geometric objects. In contrast, Carla is like someone who lacks perfect pitch for these geometric objects: she has to do further inferential work in order to determine their equations.

As a final and perhaps more dramatic example, consider Archimedean spirals. Polar coordinates make manifest the defining property of Archimedean spirals: the radius increases as a constant proportion of the polar angle, i.e. $r = a + b\theta$, for some constants $a$ and $b$. Cartesian coordinates obscure this property. In the simplest case where $r = \theta$, the corresponding Cartesian equation is $y = x\,tan(\sqrt{x^2 + y^2})$. This equation points to another interesting epistemic difference: in polar coordinates, it is possible to characterize $r$ explicitly in terms of the polar angle $\theta$. Yet, it is (seemingly) not possible to characterize $y$ explicitly in terms of the Cartesian coordinate $x$. Instead, the best we can do is represent the graph of the Archimedean spiral implicitly in Cartesian coordinates.[15]

## 5.6 Preferences, Fundamentality, and Privileging

Thus far, I have analyzed the notion of "manifest properties" in terms of what we ought to infer in a particular epistemic circumstance. Reformulating can change our epistemic circumstance, thereby changing what properties are manifest. Nevertheless, some might worry that my account does not go far enough to capture the significance of reformulations that make properties manifest. It is common for scientists and mathematicians to think that one formulation is *more fundamental* than another, but it is unclear how fundamentality could reduce to the epistemic differences that conceptualism focuses on. On this basis, a fundamentalist might claim that conceptualism owes us an account of common judgments of fundamentality. In keeping with the empiricist scruples of Chapter 1, conceptualism must provide an account of fundamentality that avoids metaphysically substantial commitments. To meet this demand, I will provide a non-metaphysical account using resources from metaethical expressivism.

### Recapping Expressivism

Metaphysicians and philosophers of science typically assume that declarative sentences about the world should be interpreted as playing a representational role. Expressivism rejects this assumption, observing that "not everything we think or say need be understood as representing the world as being some way" (Brandom 2011, p. 11). As Carnap wrote in 1934, "We have here to distinguish two functions of language, which we may call the expressive function and the representative function" (1935, p. 27).[16] Hoping to elimi-

[15]Note that we could perform another variable transformation to an $(r, \theta)$ phase space where $r$ and $\theta$ are orthogonal. In this space, the simplest Archimedian spiral $r = \theta$ is characterized by an invariant $\phi$, which corresponds to the angle measured from the $\theta$ axis. This further choice of variables makes this invariant property of Archimedean spirals *even more* manifest. In this parameterization, we effectively "unroll" the Archimedean spiral into a straight line passing at 45 degrees through the origin; we linearize the graph.

[16]See also Sellars (1958, p. 282), who denies that "the business of all non-logical concepts is to describe." Unlike Carnap, Sellars draws a more egalitarian moral, noting that "many expressions which empiricists have relegated to second-class citizenship in discourse are not *inferior*, just *different*." Carnap's 1934 lectures

nate metaphysics from analytic philosophy, Carnap proceeded to claim that "metaphysical propositions—like lyrical verses—have only an expressive function, but no representative function.…They express not so much temporary feelings as permanent emotional or volitional dispositions" (1935, p. 29). With Carnap, I agree that metaphysical statements play an expressive role. However, unlike Carnap, I am agnostic on whether this is the only role that metaphysical statements play. I will argue that we can at least make sense of physicists' judgments of fundamentality as playing a particular expressive role, regardless of whether they play a representational role as well.

Expressivism is a kind of philosophical naturalism: it explicates otherwise puzzling vocabularies in terms of non-mysterious, naturalistically acceptable ones (Price 2011). In this case, I will argue that we can understand physicists' talk about fundamentality in terms of their attitudes toward privileging some formulations or variable choices over others. Some philosophers may nevertheless hanker after something more than this kind of anthropological analysis. Namely, they may desire a representational or descriptive analysis of judgments of fundamentality. I am not inclined to stop them, although I will resist if they contend that I ought to hanker after something more as well. Brandom phrases this resistance to representationalism rather eloquently:

> If the practices themselves are all in order from a naturalistic point of view, any difficulties we might have in specifying the kind of things those engaged in the practices are talking about, how they are representing the world as being, ought to be laid at the feet of a Procrustean semantic paradigm that insists that the only model for understanding meaningfulness is a representational one. (Brandom 2011, p. 192).

In Section 3.5, I provided an expressivist account of comparative judgments of understanding. I argued that we can understand judgments of the form "X provides better understanding than Y" as expressing a mental state of *being for intellectually-preferring X to Y*. Equivalently, when someone judges a formulation X to provide better understanding than Y (for a particular problem), they endorse a set of norms that permit intellectually-preferring formulation X to formulation Y (at least for this kind of problem). In this way, we can vindicate scientists' and mathematicians' ordinary judgments about comparative understanding without having to posit metaphysically substantial facts or properties about comparative intellectual value. Structurally, this parallels how metaethical

have been reprinted in Carnap (1996 [1935]).

expressivists aim to vindicate ordinary moral (or normative) judgments without positing metaphysically substantial facts or properties about moral rightness or wrongness (or primitive ought-claims, in the case of normativity) (Blackburn 1998; Gibbard 2003).

### Expressivism about Fundamentality

Here, I propose a similar expressivist analysis of fundamentality. Indeed, there is a close connection between judgments of fundamentality and comparative judgments of understanding. To judge that X is more fundamental than Y typically entails that X provides a better understanding of some class of problems or phenomena than Y. As noted in Section 3.5, this judgment of better understanding might be aim-relative. For instance, understanding the human heart as a collection of molecules provides a better understanding relative to certain aims, but not all. A molecular understanding of the heart obscures the mechanical understanding we might achieve by describing the heart at a higher length scale, focusing on biological tissue. Still, there is a sense in which the molecular understanding is more fundamental than the biological understanding.

On my proposal, comparative judgments of fundamentality express an attitude of *being for privileging*. To judge that X is more fundamental than Y expresses a mental state of *being for privileging X to Y*. Privileging is a particularly committal form of preference. In contrast to preferring something, to privilege something typically entails that it is to be *uniquely* preferred, at least along a certain dimension. Privilege is a kind of *maximal preference*: to privilege something is to believe it is uniquely best in some regard. Whereas judgments of comparative or relative fundamentality involve privileging X over Y, we can provide a similar analysis of *absolute fundamentality*. To judge that X is absolutely fundamental is to express a mental state of being for privileging X, relative to all alternatives.

Often, when physicists and mathematicians call a fact—or entity, structure, principle, etc.—'fundamental,' they express an attitude of privileging that fact in *derivations* of other facts. Other things equal, a fact X is more fundamental than a fact Y when X figures in a derivation of Y (but not vice versa). This dimension of fundamentality is intricately connected to metaphysical notions of grounding and truth-making. At first glance, such connections might seem to pose a problem for expressivism about fundamentality. Fortunately, Barker has developed a promising expressivist approach to truth-making claims. To say that some fact(s) X (non-causally) makes it the case that Y is to express a commitment to using X to derive Y (2012, p. 273). Hence, I am optimistic that we can understand this pervasive dimension of fundamentality in terms of a pro-attitude of privileging some facts for certain derivational roles.

Overall, my expressivist analysis of fundamentality provides a simple response to the objection raised above. Conceptualism can sanction scientists' and mathematicians' ordinary judgments of relative and absolute fundamentality. Consider a physicist who claims that variables that makes the hidden hyperspherical symmetry of hydrogen manifest are more fundamental than variables that obscure this symmetry. This judgment of relative fundamentality amounts to endorsing a set of norms that permit privileging the manifest variable choice to the non-manifest variables. As we will see in Sections 5.8 and 5.9, there are a variety of reasons to privilege variables that make a symmetry manifest. Hence, we can endorse these judgments of fundamentality as a rational aspect of scientific and mathematical practice. These judgments play an important functional role in coordinating scientific and mathematical problem-solving. They help scientists converge on variable choices that have instrumental and epistemic value. We can endorse these judgments without committing ourselves to metaphysically substantial facts or properties about fundamentality. Instead, we simply focus on the non-descriptive functional roles that judgments of fundamentality perform.

**Fundamentality and Invariants**

In problem solving, we are often interested in invariant properties. Such properties allow us to characterize systems or objects across varying contexts. They provide a stable point of reference. Hence, scientists and mathematicians have a good epistemic reason to prefer expressive means that make an invariant property manifest. Expressive means that wear an invariant on their sleeves are better suited to make invariance manifest. In some sense, they *minimize* what we need to know to determine invariance. This is perhaps one reason why we often associate invariant degrees of freedom with more fundamental properties. Additionally, in physics, observables must be invariant under the symmetries of a theory. Expressive means that obscure these invariances are therefore rightly viewed as less fundamental, *ceteris paribus*: we have at least an epistemic reason to disprefer them.

In general, scientific preferences for variable choices might align with the following methodological advice: if one plans to use an epistemic dependence relation that involves a particular degree of freedom, then it is better to express that EDR in a notation that trivially wears that degree of freedom on the sleeves (i.e. is manifest to the 0th degree). Doing so will typically make a property of interest manifest, namely the property that we are using the EDR to assess. Accepting this methodological advice does not involve any further metaphysical commitments to whether this degree of freedom is fundamental in any deep sense.

Moreover, to say that a notation is particularly well-suited for expressing a particular EDR is not to say that it is uniquely suited. There could be a wide variety of expressive means that are equally well-suited for making a particular property manifest. Hence, on the account I defend, we do not have to view scientists as aiming for a single, overarching, most fundamental language. Instead, we can interpret their judgments of fundamentality as often being implicitly relativized: X is more fundamental than Y relative to a certain class of problems or a certain set of aims.

My expressivist account of fundamentality does not preclude a descriptivist or representationalist account. For all I say here, some such account could be correct. I simply claim the following: regardless of whether physicists's judgments of fundamentality amount to anything more, they at least play the functional roles that my expressivist account describes. For my purposes, it is enough to vindicate physicists's ordinary judgments of fundamentality. Unlike Carnap, I do not intend to rule out or eliminate substantial metaphysics. I have a weaker aim, namely to show that many of us can responsibly go on without such metaphysics. As Brandom notes, "a successful local expressivism about some vocabulary [e.g. fundamentality] would show that, while it might be *possible* to offer a representational semantics for that vocabulary, it is not *necessary* to do so in order to show it to be [naturalistically] legitimate" (2011, p. 195).

**An Objection from Instrumentalism**

An instrumentalist about reformulations (see Section 1.4) might object to my account of fundamentality as follows: sometimes, our overall reasons for privileging a choice of variables contains a confluence of epistemic and practical values. For instance, even though polar coordinates make manifest the invariant polar angle of a diagonal line, we might still prefer to use Cartesian coordinates to determine the equation for this line. We might prefer Cartesian coordinates for a variety of practical or idiosyncratic reasons: perhaps we dislike polar coordinates in general, or do not have a protractor, or prefer to always find the equations of straight lines using point–slope form (since this works for any straight line), etc. Surely—the instrumentalist objection continues—these reasons for privileging Cartesian coordinates have nothing to do with fundamentality.

To respond to this instrumentalist objection, it suffices to note the following: expressivism does not rely on a dispositional account of our attitudes or preferences. Instead, it relies on a fitting-attitudes account. Certain reasons are fitting for particular attitudes. For instance, an expressivist about humor does not say that jokes are funny because people laugh at them. Instead, jokes are funny when people *ought* to laugh at them. There are a wide variety of non-humor related reasons why someone might laugh at a joke. Expressivism can rightly classify those non-humor-related reasons as irrelevant to the comedic value of the joke.

Similarly, even if we dislike polar coordinates, we can still recognize that they make manifest the invariant degrees of freedom of a number of different kinds of equations. We can recognize that this gives us a reason for viewing polar coordinates as more fundamental than Cartesian coordinates for describing such equations. In other words, we recognize that reasons of personal preference are not the right kinds of reasons for judgments of fundamentality. They are not fitting to this end. Hence, an expressivist about fundamentality can agree with the instrumentalist that we often prefer certain variables for instrumental or idiosyncratic reasons. All the while, we can recognize that these instrumental reasons are not the right kinds of reasons for viewing one formulation as more fundamental than another.

## 5.7 Gauge Choices

In Lagrangian quantum field theory, gauge choices provide an illuminating example of how different formulations can make different properties manifest. In particular, different gauge choices illustrate trade-offs that can arise between different formulations. As we have already seen in the simpler context of polar vs. Cartesian coordinates, making one property manifest can come at the cost of obscuring others.

Indeed, one such trade-off arises whenever we introduce gauge degrees of freedom in the first place. On physical grounds, we know that a massless gauge field with non-zero spin—such as the photon field—has only two degrees of freedom (two physical polarization states). Nevertheless, in order to write the Lagrangian density (and hence the action) in a manifestly Lorentz invariant form, we introduce two redundant, gauge degrees of freedom. These allow us to write the gauge field $A^\mu$ as a 4-vector, supporting our syntactic criteria for manifest Lorentz covariance. As Cheung puts it, "these redundant modes are a necessary evil of manifest Lorentz covariance" (2017, p. 2). Hence, we trade-off manifest physical degrees of freedom for manifest Lorentz invariance. Why do physicists so often make this trade? By enforcing Lorentz invariance in the Lagrangian density, we massively constrain the space of possible interaction terms. This strategy has tremendous epistemic power for theory construction.

Here, I will focus on comparing two families of gauge choices: i) manifestly Lorentz covariant gauges vs. ii) manifestly unitary gauges. As their names indicate, they respectively make manifest the properties of Lorentz covariance and unitarity. They also each obscure the property that the other makes manifest, illustrating a trade-off. We can understand these gauges as having a symbiotic relationship: to prove that a quantum field theory is unitary, it is best to use a manifestly unitary gauge. In contrast, for most other calculations, it is best to use a manifestly Lorentz covariant gauge, since they tend to simplify calculations (Siegel 2005, p. 30). Fortunately, since these are compatible formulations, we are not forced to choose between them. Gauge choices like these provide evidence that we can understand particle physicists as exploiting different variable choices in different contexts, rather than as aiming at a single fundamental language for describing scattering processes. As Siegel notes, we have "different gauges for different uses" (2005, p. 13). Against Maudlin's (2018, pp. 14, 16) methodological recommendations, I deny any need to interpret these gauge choices as leading to competing or rival ontologies.[17]

Before delving into these gauge choices, a few remarks on "unitarity," i.e. the property of being unitary. A quantum field theory is unitary provided that it satisfies two conditions: i) all probabilities for scattering processes are non-negative and ii) probability is conserved, i.e. the probabilities of all possible processes sum to one. This second con-

[17]It is only in contexts where we view the gauge field $A^\mu$ as being a calculational device—such as some interpretations of classical electromagnetism—that Maudlin sanctions interpreting different gauge choices as leading to compatible formulations. By also interpreting different gauge choices in quantum field theory as leading to compatible formulations, I violate Maudlin's interpretive norms. Maudlin may view me as being afflicted with "the attitude of the engineer rather than the natural philosopher" (2018, p. 6). So much the worse for the natural philosopher, say I!

dition amounts to the Hamiltonian being Hermitian, i.e. $H^\dagger = H$ (Siegel 2005, p. 355).[18] The first condition requires that the inner product on Hilbert space is positive definite. This condition is more difficult to check, and it is the one that unitary gauges help make manifest.

### Manifestly Lorentz Covariant Gauges

Lorenz gauge provides a constraint on the gauge field $A^\mu$ that is manifestly Lorentz covariant: $\partial_\mu A^\mu = 0$. By constraining the gauge field in a manifestly covariant manner, we preserve the manifest covariance of those expressions that were already manifestly covariant before we imposed this constraint.

In Lagrangian quantum field theory, we generalize Lorenz gauge to the family of $R_\xi$ gauges. To gauge-fix in this manner, we add a manifestly Lorentz invariant term to the Lagrangian: $-\frac{(\partial_\mu A^\mu)^2}{2\xi}$. Different values of the parameter $\xi$ result in different gauge-fixings. Provided that the Lagrangian is already manifestly Lorentz invariant, the additional $R_\xi$ term preserves this manifest Lorentz invariance.

Two common $R_\xi$ gauges are Landau gauge and Feynman–'t Hooft gauge, which set $\xi$ equal to zero and one, respectively. Landau gauge recovers Lorenz gauge in the limit as $\xi$ goes to zero. Feynman–'t Hooft gauge ($\xi = 1$) is particularly advantageous for explicit calculations because it tends to give the simplest form for the propagator terms. In general, propagators in $R_\xi$ gauge take the form $2[\frac{\eta_{ab}}{p^2} + (\xi - 1)\frac{p_a p_b}{(p^4)}]$ (Siegel 2005, p. 389). Clearly, setting $\xi = 1$ results in the simplest propagator term: $2\frac{\eta_{ab}}{p^2}$. These gauges also have the advantage of easily generalizing from Abelian to non-Abelian symmetry groups.

According to Siegel, the $R_\xi$ gauges "manifest as many global invariances as possible" (2005, p. 386). By preserving manifest Lorentz covariance, the $R_\xi$ gauges trivialize the preservation of these space-time symmetries. In other words, it becomes unnecessary to explicitly calculate that these symmetries are preserved. Instead, the expressions continue to wear these properties on the sleeves. Wearing properties on the sleeve has non-practical epistemic value (in addition to any practical value it might have as well). The symmetry properties of these expressions become more intelligible, at least on account of becoming less surprising.

[18] Alternatively, a unitary quantum field theory has a unitary evolution operator $U$, where this means that $U(t_2,t_1)^\dagger U(t_2,t_1) = I$. This requirement amounts to the conservation of probabilities (Siegel 2005, p. 298).

### Manifestly Unitary Gauges

I turn now to manifestly unitary gauges. These include light cone and space cone gauge. Not only do these gauges make unitarity manifest, but also they eliminate unphysical degrees of freedom (such as those coming from ghost fields). I will focus in particular on light cone gauge, which according to Siegel is "the simplest for analyzing physical degrees of freedom, since the maximum number of degrees of freedom is eliminated" (2005, p. 211).

Light cone gauge relies on a light cone basis, which uses a different basis for the metric $\eta^{ab}$. Rather than focus on the $A^0$ and $A^1$ components of the gauge field $A^\mu$, we focus on their linear combinations, calling the resulting components $A^+$ and $A^-$, where $A^\pm = \frac{1}{\sqrt{2}}(A^0 \pm A^1)$. To work in the light cone gauge, we first fix one degree of freedom by setting $A^+ = 0$. To eliminate the second gauge degree of freedom, we introduce the component $A^-$ as an auxiliary field in the Lagrangian density $\mathcal{L}$, ultimately eliminating it (Siegel 2005, p. 210). We thereby reduce the four degrees of freedom in $A^\mu$ to two, representing the actual physical degrees of freedom of the gauge field.

As mentioned above, unitarity requires that the inner product on Hilbert space be positive definite (this amounts to a requirement that the energy is positive). The sign of the energy is intimately connected with the sign of the kinetic term in the Lagrangian density. By eliminating unphysical degrees of freedom, light cone gauge sets up a simple correspondence between the sign of the kinetic terms and unitarity. Hence, one can 'read off' unitarity from the Lagrangian density when it is written in light cone gauge. We simply require that boson fields have a negative kinetic energy term while fermion fields have a positive one (Siegel 2005, p. 357). In this way, the Lagrangian density in light cone gauge wears unitarity on the sleeves, thereby making it manifest.

### Trade-offs and Fundamentality

These two families of gauge choices illustrate the kinds of trade-offs that frequently arise when we change variables. On the one hand, manifestly Lorentz covariant gauges make manifest a (contextually) fundamental symmetry. Nevertheless, they obscure both unitarity and some physical degrees of freedom. On the other hand, manifestly unitary gauges obscure Lorentz invariance, despite eliminating a greater number of unphysical degrees of freedom. A fundamentalist might be inclined to weigh these trade-offs in an attempt to determine which choice of variables is more fundamental *tout court*, or which leads to a more virtuous physical theory (perhaps Maudlin (2018, p. 20) would make this recommendation). I am pessimistic about the prospects of this approach. Clearly, we can use both gauge choices in different contexts. Insofar as a physicist might be inclined to say that one gauge choice is more fundamental in a particular context, we can understand them as expressing an attitude of being for privileging this gauge choice in such contexts (Section 5.6).

Perhaps a fundamentalist might reason as follows: it is epistemically possible for there to be a variable choice that makes manifest all of these properties, with none of the drawbacks. Such a choice of variables or gauge would make manifest i) Lorentz covariance, ii) unitarity, and iii) eliminate unphysical degrees of freedom. If we had such a choice, it would be more fundamental than either of the gauge choices discussed above. Perhaps then, physicists or metaphysicians should be aiming for such a choice of variables. In many ways, spinor–helicity variables accomplish some of these aims. Yet, they also introduce trade-offs of their own. In particular, spinor–helicity variables i) obscure the property of locality and ii) introduce unphysical complex momenta (Elvang and Huang 2015, p. 61). This provides grounds for pessimism that physics will in general arrive at a choice of variables that make manifest all fundamental properties. At least sometimes, when we make one physically significant property manifest, it comes at the cost of obscuring others.[19] Of course, I have looked at only a small set of cases. Nevertheless, these examples motivate a more extensive inductive argument (for future work) that would parallel the Pessimistic Meta-Induction against scientific realism.

## 5.8 Manifest vs. Hidden Symmetries of Hydrogen

The symmetries of the hydrogen atom provide a striking contrast between manifest vs. hidden properties. In elementary presentations, the hydrogen atom has a manifest spherical symmetry but a hidden hyperspherical symmetry. My account of manifest vs. hidden properties from Section 5.2 makes these claims precise. There is an epistemic circumstance where i) one ought to infer that hydrogen has spherical symmetry but where ii) it seems impermissible to infer that hydrogen has a larger hyperspherical symmetry (at least in this

[19] I thank Henriette Elvang for encouraging me to weaken some more sweeping claims in favor of pessimism.

epistemic circumstance). Before making this further inference, more inferential work is required: one must transform the given epistemic circumstance into a different one.

Many models exist for the hydrogen atom, but not all of them exhibit a hidden hyperspherical symmetry. Hence, I will focus on a model for a nonrelativistic, spinless hydrogen atom. This model was the first to be worked out after the advent of quantum mechanics (Pauli 1926). Despite neglecting relativity and electron spin, this simple model provides a robust first-order approximation of the hydrogen atom's energy-level spectrum.[20]

### Manifest Spherical Symmetry

In nonrelativistic quantum mechanics, we determine properties of a system by analyzing its Hamiltonian, often in conjunction with the Schrödinger equation. For a nonrelativistic, spinless hydrogen atom, the Hamiltonian consists of two terms: a kinetic term for a free particle and a potential energy term given by Coulomb's law of electrostatics:[21]

$$H = \frac{\mathbf{p}^2}{2\mu} + V(x,y,z) = -\frac{h^2}{8\pi^2\mu}\nabla^2 - \frac{e^2}{4\pi\varepsilon_0 r} \tag{5.8.1}$$

On its own, the Hamiltonian (5.8.1) makes manifest that the hydrogen atom has spherical symmetry. This is because both the kinetic and potential terms are manifestly invariant under arbitrary rotations in three-dimensional Euclidean space, entailing that $H$ is likewise spherically symmetric (since a sum of spherically symmetric terms is spherically symmetric). Clearly, the various constant terms in the expression are invariant under three-dimensional rotations, so all we need to do is check the invariance of the non-constant functions, namely $\nabla^2$ and $1/r$. To see that the kinetic term is spherically symmetric, it suffices to unpack the $\nabla^2$ operator, known as the Laplacian: $\nabla^2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}$. With each Cartesian coordinate on equal footing, this term is invariant under three-dimensional rotations. Turning to the potential term, the function $1/r = 1/\sqrt{x^2+y^2+z^2}$ again places each of the three Cartesian coordinates on equal footing, so its rotational invariance is manifest. Since each term is rotationally invariant, so is the Hamiltonian. It thus has at least the symmetry of the group of proper rotations in three-dimensional

[20] More sophisticated treatments using the Dirac equation and quantum electrodynamics later accounted for higher-order features of the hydrogen spectrum. However, they break the special "dynamical" symmetry of this simple model.

[21] Here, $\mu$ is the reduced electron mass $\frac{m_e m_p}{m_e + m_p}$, a function of the electron and proton masses.

Euclidean space, known as the *special orthogonal group in three-dimensions*, $SO(3)$.

Based on the account in Section 5.2, to say that the Hamiltonian $H$ has manifest spherical symmetry is to say that we ought to infer that it is spherically symmetric. More precisely, provided that we implement the problem-solving plan described above, we are in an epistemic circumstance where we ought to infer $SO(3)$ symmetry. Since hydrogen has a spherically symmetric Hamiltonian, it follows that the hydrogen atom has at least this symmetry.[22] If we do not make this inference based on the reasoning above, then we have made an epistemic mistake. We would be doing something epistemically deficient.

Indeed, the foregoing analysis shows that we can say something even stronger: the Hamiltonian in (5.8.1) *wears its spherical symmetry on the sleeves.* The property is not only manifest, but it is made manifest solely on the basis of features of equation (5.8.1) that are already manifest, i.e. manifest before we implement the problem-solving plan detailed above. These already-manifest properties include the placement and identity of the various terms in the expression. On the basis of these syntactical properties, we ought to infer that the constant terms, $\nabla^2$, and $1/r$ are all spherically invariant.[23] On the basis of these inferences, we then ought to infer that $H$ is spherically invariant as well. $SO(3)$ symmetry is a sleeve property of the hydrogen atom Hamiltonian.

**Hidden Hyperspherical Symmetry**

We can now contrast the manifest status of $SO(3)$ symmetry with the completely different epistemic situation for hydrogen's hidden symmetry. This hidden symmetry is associated with special features of the two-body problem with a $1/r$-potential, leading many physicists to deem it a "dynamical symmetry"—in contrast with "geometrical symmetries" that arise from spacetime symmetries.[24] It turns out that this simple model of the hydrogen

[22]In this context, the symmetry group of a system is defined as the group of operators that commute with its Hamiltonian. Since the Hamiltonian is invariant under three-dimensional rotations, all of these operators commute with $H$, i.e. $[H,R] = HR - RH = H - H = 0$, for any $R \in SO(3)$.

[23]As shown above, the spherical invariance of these terms becomes manifest when we implement a problem-solving plan for them, such as writing out '$1/r$' explicitly as a function of Cartesian coordinates. This example thereby illustrates the gradated nature of manifest properties.

[24]In this context, a dynamical symmetry refers to a symmetry that is associated with the particular form and nature of the dynamics, e.g. the particular form of a force law, number of interacting subsystems, or energy state of the system (bound or scattering). Note that this is a narrower notion of "dynamical symmetry" than that commonly found in the philosophy of physics literature, where *dynamical symmetries* are those that leave the model's equations of motion invariant. In this broader sense of dynamical symmetry, hydrogen's $SO(3)$ symmetry is also dynamical.

atom has a much larger symmetry group, namely the symmetry of a four-dimensional Euclidean hypersphere. Formally, this group is known as the special orthogonal group in four dimensions, denoted by '$SO(4)$.'

Following the account of Section 5.2, we can at least say that this hyperspherical symmetry is *not manifest*: when presented merely with the Hamiltonian in equation (5.8.1), it is not the case that we ought to infer that this Hamiltonian has hyperspherical symmetry. We do not make an epistemic mistake if we fail to make this inference. Thus, there is an epistemic circumstance $C$ where i) we ought to infer that hydrogen has $SO(3)$ symmetry, but ii) it is not the case that we ought to infer that it has $SO(4)$ symmetry.

Indeed, I am tempted to assert a stronger claim: not only is the hyperspherical symmetry not manifest in this epistemic circumstance, it is hidden. In other words, if we were to infer that $H$ has hyperspherical symmetry solely on the basis of this epistemic circumstance, then we would make an epistemically impermissible inference. We would be jumping to conclusions in an irrational manner.[25] In order to license the inference that hydrogen has hyperspherical symmetry, more epistemic work is required. We must transform our epistemic circumstance into one where we are rationally permitted to infer this symmetry.

It is precisely this transformation of epistemic circumstances that Fock undertook in his analysis of the hydrogen atom (1935b).[26] By changing variables to momentum space, Fock was able to make manifest the hyperspherical symmetry of hydrogen. Schematically, Fock's argument proceeds as follows: we write the integral form of the Schrödinger equation in momentum space. Using a stereographic projection from the three-sphere $S^3$ to Euclidean three-space $\mathbb{R}^3$, we then demonstrate that this equation is equivalent to an integral equation for the four-dimensional spherical harmonics. We thereby see that the four-dimensional spherical harmonics are solutions to the hydrogen atom's Schrödinger equation. Since these spherical harmonics have $SO(4)$ symmetry, so must the hydrogen atom. Hence, by the end of this argument, the hyperspherical symmetry of hydrogen has become manifest (although it is plausibly not worn on the sleeves of a corresponding expression). This schematic discussion suffices for my philosophical aims here. For the

[25] Compare Field's (2018, p. 5) discussion (stemming from Boghossian) of someone applying an inference rule that—although sound—has not yet been demonstrated to be sound. Such a person would plausibly strike us as being irrational, even if their inference follows a reliable pattern.

[26] See Fock (2005) for an English translation. See also Fock (1935a).

interested reader, I provide more details about Fock's argument below. Less interested readers can happily skip ahead to Section 5.9.

**Fock's Argument, in more Detail**

Using the Hamiltonian in equation (5.8.1), we can write the time-independent Schrödinger equation $H\psi = E\psi$ for the hydrogen atom, where $E$ is an energy eigenvalue of the hydrogen wavefunction $\psi$. This results in the following equation, written in position space:

$$-\frac{h^2}{8\pi^2\mu}\nabla^2\psi(x,y,z) - \frac{e^2/4\pi\varepsilon_0}{\sqrt{x^2+y^2+z^2}}\psi(x,y,z) = E\psi(x,y,z) \tag{5.8.2}$$

Fock performs a Fourier transform on this Schrödinger equation, expressing it in momentum space:

$$\frac{\hbar^2}{2m}|p|^2\,\psi(p) - e^2\sqrt{\frac{2}{\pi}}\int_{\mathbb{R}^3}\frac{\psi(p')dp'}{|p-p'|^2} = E\psi(p) \tag{5.8.3}$$

The form of this equation motivated Fock to consider a stereographic projection from $S^3$ to $\mathbb{R}^3$. According to McIntosh, "In this form, the kernel can be recognized as the Jacobian determinant for a stereographic projection from the surface of a four-dimensional sphere to three dimensions, which in turn suggests writing the Schrödinger equation in terms of angular variables on the hyperspherical surface" (1971, p. 81). Fock denotes these angular variables as $(\alpha,\theta,\phi)$, and introduces a function $\Psi(\alpha,\theta,\phi)$ defined on the hypersphere. $\Psi(\alpha,\theta,\phi)$ depends as well on the momentum and energy of the atomic state. Using this function, he expresses the Schrödinger equation on the hyperspherical surface as follows:

$$\Psi(\alpha,\theta,\phi) = \frac{\lambda}{2\pi^2}\int\frac{\Psi(\alpha',\theta',\phi')d\Omega'}{4sin^2(\omega/2)} \tag{5.8.4}$$

Here, $\lambda = \frac{me^2}{h\sqrt{-2mE}}$ and $d\Omega$ is the surface element for the 3-sphere. The term $4sin^2(\omega/2)$ in the denominator of the integrand represents the square of the distance between the two points $(\alpha,\theta,\phi)$ and $(\alpha',\theta',\phi')$ on the 3-sphere (hence, $\omega$ is the arclength of the great circle that connects the two points) (Fock 2005, p. 288).

Fock then compares this reformulation of the Schrödinger equation to the integral equation for the four-dimensional spherical harmonics:

$$r^{n-1}\Psi_n(\alpha,\theta,\phi) = \frac{n}{2\pi^2}\int \frac{\Psi_n(\alpha',\theta',\phi')d\Omega'}{1-2rcos(\omega)+r^2} \tag{5.8.5}$$

Setting $\lambda = n$ and $r = 1$, we recover the same form as the Schrödinger equation (5.8.4). McIntosh interprets this case as being "the Poisson kernel for a hyperspherical surface harmonic in the degenerate case in which the field point has fallen onto the surface," where $r = 1$ specifies the surface (1971, p. 81). Physically, the integer $n$ is the principal quantum number, labeling the hydrogen atom's energy levels. Due to this correspondence between the two equations, we see that the hydrogen atom wavefunctions can be expressed in terms of the hyperspherical harmonics. Hence, any symmetry of these harmonics is a symmetry of the hydrogen atom wavefunctions, and thus of the hydrogen atom itself.[27]

Fock summarizes the conclusion of his argument as follows:

> Thus we have shown that the Schrödinger equation (5.8.3) or (5.8.4) can be solved with four-dimensional spherical harmonic functions. At the same time the transformation group of the Schrödinger equation has been found: this group is obviously identical to the four-dimensional rotation group. (Fock 2005, p. 289)

Alternatively, we can interpret Fock as having constructed a representation of the group $SO(4)$ on the phase space of the hydrogen atom (namely, the space of square integrable functions on $\mathbb{R}^3$). Fock implicitly shows that this representation commutes with the Hamiltonian for hydrogen. This entails that the hydrogen atom has hyperspherical symmetry (Singer 2005, p. 283).

More precisely, this symmetry applies only to *bound states* of hydrogen, namely those where the electron has negative potential energy. These states constitute the discrete or 'point' spectum for hydrogen. If the electron acquires enough energy, it enters a scattering state (positive potential energy), leading to a continuous spectrum. In this case, the symmetry is that of the Lorentz group, and the geometrical interpretation relies on a hyperboloid rather than a hypersphere (McIntosh 1971, p. 81; Fock 2005, p. 292).

Note that Fock's momentum space representation (5.8.3) of the hydrogen atom Schrödinger equation plausibly does *not* wear the hyperspherical symmetry on its sleeves. Hence, although we have made the symmetry manifest by the end of the

[27]Note that the four-dimensional hyperspherical harmonics have hyperspherical symmetry in virtue of being the angular part of the solutions to the Laplace equation in four dimensions.

derivation (one ought to infer that the system has the symmetry), we have not done so by making the symmetry into a sleeve property of a corresponding expression.

## 5.9 Hidden Symmetries in $\mathcal{N} = 4$ super Yang–Mills Theory

Precision calculations for predictions at the Large Hadron Collider increasingly require calculations at third-order or higher in perturbation theory. These calculations are necessary to gain a better theoretical understanding of background processes. Without theoretical knowledge of the background, it is impossible to isolate new physics from already understood processes. This task is challenging largely because of how quickly the number of terms grows in perturbation theory. To manage this computational complexity, physicists have had to repeatedly reformulate their calculational techniques. Feynman diagrams provide one such reformulation, but these techniques become infeasible for scattering more than a few particles, due to the rapid growth of diagrams. More recently, physicists have reformulated pertubation theory calculations using spinor–helicity variables, in a method known as *on-shell recursion*. At tree-level, this method factorizes amplitudes involving $n$-many particles into products of scattering amplitudes with fewer than $n$-particles. At loop-level, on-shell recursion takes advantage of unitarity cuts to factorize loop amplitudes into lower order amplitudes. In this way, we arrive at general recursion relations for computing higher-order scattering processes.[28]

On-shell recursion illustrates how different choices of variables can make certain properties or patterns manifest. For instance, an elegant relationship known as the Parke–Taylor formula requires hundreds of pages to prove using Feynman diagrams but only a three-page inductive proof using the on-shell formulation. Progress in particle physics often comes from figuring out how to re-package perturbation series into ever more convenient forms, where otherwise-mysterious cancellations become clear. As noted in Section 5.1, some metaphysicians might be tempted to describe these examples as the result of finding a more fundamental language. In contrast, I agree with Woodward (2016, p. 1056) that metaphysical appeals to joint-carving do not give us a satisfying account of the relevant epistemological issues. The challenge is to understand how certain variable choices can make previously mysterious calculational patterns and cancellations intelligible.

[28]For background, see Henn and Plefka (2014), Dixon (2016), and Cheung (2017).

On-shell methods for scattering amplitudes illuminate a hidden symmetry that the tree-level superamplitudes possess in $\mathcal{N}=4$ super Yang–Mills theory. This theory has an "obvious" superconformal symmetry $SU(2,2|4)$ that leaves its superamplitudes invariant.[29] Additionally, the tree-level superamplitudes of this theory possess a non-obvious *dual superconformal symmetry*. This hidden symmetry is also expressed by the symmetry group $SU(2,2|4)$, but now acting on a different set of variables defined in a different space than ordinary momentum variables (Elvang and Huang 2015, p. 95). Accounting for the intellectual significance of this hidden symmetry has numerous parallels to interpreting the hidden $SO(4)$ symmetry of the nonrelativistic hydrogen atom. As we saw in Section 5.8, an elementary presentation of the Hamiltonian for hydrogen does not make this hyperspherical symmetry manifest, although it does wear an "obvious" $SO(3)$ symmetry on the sleeves. Furthermore, this hidden symmetry is made manifest by moving to momentum variables. Similarly, in $\mathcal{N}=4$ super Yang–Mills theory, the Lagrangian does not make manifest the hidden dual superconformal symmetry of the tree-level amplitudes. This hidden symmetry is made manifest by doing a series of variable changes, first moving to twistor space, then to a dual space, and finally to momentum twistor space. This section describes this series of variable transformations and the epistemic advantages we gain along the way.

In both examples, we can account for the intellectual significance of hidden symmetry in terms of epistemic dependence relations: moving to variables that make the symmetry manifest changes what it suffices to know to figure out if a given mathematical expression possesses the relevant symmetry. As discussed at the end of Section 5.4.1, by constructing objects that possess manifest dual superconformal symmetry, one can immediately infer that a more complicated object constructed from these invariant pieces also possesses this symmetry. There is surely part of this variable change that is merely convenient, but the change in epistemic dependence relations is also intellectually significant.

The main method for showing that superamplitudes possess a given symmetry is to show that the generators of that symmetry annihilate the superamplitudes. For instance,

[29] This symmetry group comprises a conformal part $SU(2,2)$ and an $R$-symmetry-part $SU(4)$. $SU(2,2)$ consists of $4\times 4$ complex matrices of determinant one that preserve a Hermitian quadratic form of signature $(-1,-1,1,1)$. It is locally isomorphic to the conformal group $SO(2,4)$ of spacetime. The $R$-symmetry $SU(4)$ acts on the supersymmetry generators $Q^A$ and $Q^\dagger_A$, where the index $A$ ranges from one to four. These four 'supercharges' generate the supersymmetry transformations that transform bosons into fermions and vice versa.

tree-level superamplitudes possess Poincare symmetry because the ten generators of the Poincare group (four translations and six rotations/boosts) annihilate these amplitudes (Elvang and Huang 2015, p. 96). This holds for the super-Poincare group as well, which adds 16 fermionic supersymmetry generators $Q^{Aa}$ and $\tilde{Q}^{\dot{a}}_{A}$. To show invariance under the superconformal group, the proof focuses on 16 additional fermionic conformal supersymmetry generators $S_{Aa}$ and $\tilde{S}^{A}_{\dot{a}}$ along with properties of the momentum delta function and supermomentum Grassmann delta function (Elvang and Huang 2015, p. 99).

### Changing to Twistor Variables

Representing the 62 superconformal symmetry generators of the graded Lie algebra $su(2,2|4)$ in spinor-helicity variables fails to treat these generators on equal footing. For instance, the translation generator has no derivative terms, the rotation/boost generators have one derivative term, and the conformal boost has two derivatives (Elvang and Huang 2015, p. 97). A desire to place these generators on equal footing with regards to derivative terms motivates the first change of variables. By moving to twistor variables, it is possible to provide a representation of these generators where every generator is a 1-derivative operator, which means that each has been linearized (Elvang and Huang 2015, p. 100). Just as the hydrogen atom case involves a Fourier transform from position space to momentum space, this variable change involves a Fourier transform from angle spinor variables to twistor variables. The resulting variables $\mathcal{W}^{A}_{i}$ are called supertwistors, and they consist of a triple of a square spinor, the Fourier transform of an angle spinor, and a Grassmann variable. This leads to a compact expression for every generator of the superconformal algebra where every generator is treated uniformly (Elvang and Huang 2015, p. 100). Furthermore, since the supertwistors scale homogenously under little group transformations, the resulting expression for the symmetry generators are invariant under this transformation.[30] This leads to a projective characterization of the twistors and supertwistors. The bosonic twistor part can be defined as a point in complex projective three space $\mathbb{CP}^3$. The supertwistors are points in $\mathbb{CP}^{3|4}$ space.

Changing variables to twistor space leads to a geometric interpretation of $n$-gluon tree-level amplitudes. It turns out that a tree-level gluon amplitude with $q$-many posi-

[30] In this context, the little group is the subgroup of the Poincare group that leaves the 4-momentum of a particle invariant. For massless particles, this is the two-dimensional Euclidean group $ISO(2)$, comprising translations in space and rotations around the direction of motion.

tive helicity gluons corresponds to a set of twistor points on a $(q-1)$-dimensional curve in bosonic twistor space $\mathbb{CP}^3$. For instance, anti-MHV amplitudes have two positive helicity gluons, so they correspond to a 1-dimensional curve. Hence, the amplitude itself comprises $n$ twistors that lie on the same line in $\mathbb{CP}^3$ (Elvang and Huang 2015, p. 101).

Using Dirac's embedding formalism, one can provide an interpretation of twistors as a projective representation of spacetime points and null-lines. In the embedding formalism, the conformal group $SO(2,4)$ is realized as the Lorentz group of a six dimensional spacetime with metric $(-,-,+,+,+,+)$. Twistors are then defined as spinors on a conformal 4-dimensional subspace that satisfies a null condition $X \cdot X = 0$ and projectively identifies the 6-dimensional vector $X$ with any scalar multiple $rX$. Each point $X$ in this four-dimensional subspace is fixed by two twistor variables $W_i$ and $W_j$. In other words, a line in twistor space corresponds to a point in the four-dimensional embedded spacetime. Conversely, any of two (six-dimensional) spacetime points $X_i$ and $X_j$ define a null-line, and they share the same twistor (since each twistor is identified with any scalar multiple of itself—resulting in twistor space being again $\mathbb{CP}^3$). Thus, a null-line in spacetime corresponds to a point in twistor space. In this way, twistor space is dual to the 4-dimensional embedded spacetime (dual in the same sense that lines and points are dual to each other in projective geometry).

### Changing to Dual Coordinates

There are a few expressive disadvantages of twistor variables that motivate yet another variable change (taking us closer to making the hidden dual superconformal symmetry manifest). In the twistor variables, the translation generators of the Poincare group do not have a linear action on spinor variables. This means that the spinor variables are not invariant under translation, and hence both momentum and supermomentum are not automatically conserved in the supertwistor formalism. Instead, momentum and supermomentum conservation are enforced using delta functions (Elvang and Huang 2015, p. 103). Just as one of the motivations for spinor–helicity variables is to automatically enforce the on-shell condition (to "trivialize" this condition), the next variable change is motivated by a desire to automatically enforce conservation of momentum. We do this by interpreting momentum conservation geometrically, as a closed, convex contour, represented by a polygon. The momenta 4-vectors are directed edges of this $n$-sided polygon. The closure condition provides a geometric interpretation of the $n$-many momenta summing to zero.

A key step in characterizing the hidden symmetry is to move from the edges of this polygon (the momenta 4-vectors) to the dual notion, i.e. the points that define the vertices/cusps of the polygon. These dual coordinates $y_i^\mu$ define a dual space that although consisting of dual momentum variables is not itself characterized using spacetime coordinates. In this dual space, momentum conservation for scattering $n$-many particles is enforced by requiring that the $(n+1)$-th cusp $y_{n+1}$ is the same as the first cusp of the polygon $y_1$, i.e. by requiring that the cusps are periodic (Elvang and Huang 2015, p. 103). Unlike the 4-momenta variables, these dual coordinates are invariant under translations. They thereby wear momentum conservation on their sleeves.

We proceed to re-express previous tree-level amplitude expressions using these dual space coordinates (and corresponding dual space coordinates for fermion variables). Since these amplitudes are now defined in dual space, it is possible to investigate a new class of symmetries, namely those encapsulated by dual superconformal symmetry. In this analysis, the dual inversion operator $I$ plays a special role because the conformal boost generators $\mathcal{K}^\mu$ can be defined as intertwined with the translation operator by inversion: $\mathcal{K}^\mu = I\mathcal{P}^\mu I$. Since the dual coordinates are invariant under dual translation, this relationship makes it easy to see how the dual coordinates transform under other symmetry generators of the dual superconformal group. Ultimately, using the super-BCFW recursion relations re-expressed in these variables, it can be shown that all of the tree-level superamplitudes of $\mathcal{N} = 4$ SYM are invariant under dual superconformal symmetry. Even though this symmetry group is the same as that for regular superconformal symmetry, the symmetries are distinct. For instance, the tree amplitudes for gluon scattering in pure Yang–Mills theory are conformally invariant but not invariant under the corresponding symmetries of the dual conformal group (Elvang and Huang 2015, p. 105).

The two superconformal groups (ordinary and dual) can be combined into an even larger symmetry group known as the Yangian. This group has a countably infinite-dimensional algebra, where the lowest level generators correspond to the generators of the ordinary superconformal group. Since the tree-level superamplitudes are invariant under both ordinary and dual superconformal symmetry, they are ultimately invariant under the Yangian as well, manifesting an even larger hidden symmetry (Elvang and Huang 2015, p. 106).

### Changing to Momentum Twistors

Despite demonstrating the dual superconformal symmetry by using dual coordinates, these coordinates are not ideal for expressing this symmetry. They do not themselves transform covariantly under the symmetry group (Elvang and Huang 2015, p. 107). Consequently, the resulting expressions for tree-level superamplitudes also do not wear this dual superconformal symmetry on their sleeves in dual coordinates. This is what ultimately motivates moving to *momentum twistors*.

Just as the twistor variables are geometrically dual to spacetime coordinates, the momentum twistors are geometrically dual to the dual coordinates $y_i^\mu$. This means that a momentum twistor corresponds to a null-line in the dual $y$-space, and a point in the $y$-space corresponds to a line in the momentum twistor space. Furthermore, these momentum twistors transform as spinors, i.e. they have spinor indices. For convenience, we will call the momentum twistor space $Z$-space. The momentum twistor variables $Z_i^I$ transform linearly under every transformation of the dual conformal group $SU(2,2)$, leading to a uniform and compact expression for the generators of this group (Elvang and Huang 2015, p. 108).

To re-express the amplitudes in a way that is manifestly invariant under dual conformal transformations, we form an invariant object out of the momentum twistor variables by contracting four of them with the Levi-Civita tensor for $SU(2,2)$. This leads to an invariant object called the 4-bracket, allowing us to re-express both the on-shell propagators and the tree-level amplitudes. Although the 4-bracket is convenient due to its symmetry properties, it is more than merely convenient: by building further objects (such as amplitudes) out of 4-brackets, it follows that these objects inherit the symmetry properties of dual conformal invariance. This is an instance of an epistemic dependence relation: to know that a resulting expression is invariant under the dual conformal group, it suffices to know that it is built out of component parts that are invariant.

By adding a corresponding Grassmann-variable to the momentum twistors, one forms momentum supertwistors, which make the dual superconformal symmetry manifest (Elvang and Huang 2015, p. 110). Here is a summary of the methodological upshot of all of these variable changes:

> Starting with the simple observation that momentum conservation is imposed in a rather ad hoc fashion, we introduced the auxiliary variables $y_i$ such that momentum

> conservation is encoded in a geometric fashion. This led us to the realization of a new symmetry of the tree amplitude for $\mathcal{N}=4$ SYM, namely superconformal symmetry in the dual space $y_i$. The new symmetry set us on a journey to search for new variables, the momentum (super)twistors, that linearize the transformation rules. This culminated in the simple symmetric form of the $n$-point NMHV tree superamplitude. (Elvang and Huang 2015, p. 111)

Finally, the momentum twistors and momentum supertwistors have a further expressive property lacked by the dual coordinates in $y$-space. Although the $y$-space coordinates trivialize momentum conservation, they are nevertheless forced by hand to obey an algebraic constraint: $(y_i - y_{i+1})^2 = 0$. This enforces the on-shell momentum condition for the 4-momentum $p_i$. In contrast, the $Z$-coordinates are not subject to any analogous constraint. These coordinates are thereby defined freely in $\mathbb{CP}^3$. Working in this space of free $Z$-coordinates, we can study scattering amplitudes for $n$-many particles by picking *any* set of $n$-many points $Z_i$. To represent a scattering process, these points must ultimately form a closed contour, which means that each line (edge) is characterized by connecting subsequent points, i.e. $(Z_i, Z_{i+1})$. Due to the projectively dual relationship between $y$-space and $Z$-space, each of these lines $(Z_i, Z_{i+1})$ corresponds to a dual coordinate $y_i$. The fact that the contour is closed simply means that the $n$th line is $(Z_1, Z_n)$, which entails the periodicity condition in dual coordinate space, i.e. that $y_{n+1} = y_1$. Recall that this periodicity condition simply means that momentum is conserved. In this way, our construction of a representation for scattering amplitudes in $Z$-space automatically enforces momentum conservation. Furthermore, the mapping of lines in $Z$-space to points in $y$-space forces adjacent $y_i$ and $y_{i+1}$ coordinates to obey an *incidence relation* that forces these adjacent $y$-coordinates to be null-separated. Since these adjacent coordinates are null-separated, the associated 4-momenta $p_i$ are on-shell. In this way, the momentum twistor construction also automatically enforces that the represented scattering process is on-shell. This is a key difference with the $y$-space formalism itself, where the on-shell condition had to be enforced by hand (by requiring that adjacent $y$-coordinates be null-separated). Similar remarks apply for the momentum supertwistors (Elvang and Huang 2015, p. 112).

Moreover, momentum twistors provide a geometric interpretation of the propagators. In the dual space coordinates, propagators are expressed as $1/y_{ij}^2$, and a propagator is on-shell when $y_{ij}^2 = 0$. Using the aforementioned 4-bracket (which is a dual conformal invariant expressed in terms of four momentum twistors), the on-shell condition is re-expressed as requiring that the 4-bracket equals zero. Algebraically, this means that the four momentum twistors defining the 4-bracket are linearly dependent. Geometrically, this means that these four twistors belong to the same plane in $\mathbb{CP}^3$. This interpretation can be extended further by recasting propagator poles $y_{ij}^2 = 0$ as the intersections between certain lines and planes in momentum twistor space (Elvang and Huang 2015, p. 113).

To phrase this all more starkly: the reformulation using momentum twistors has enabled kinematic constraints (momentum conservation, on-shell momenta, and propagator poles) that were previously expressed *algebraically* (i.e. as solutions to equations) to be expressed *geometrically* (i.e. in terms of the intersections of lines and planes at certain points in momentum twistor space). This is yet another illustration of a difference in epistemic dependence relations. Rather than needing to know that a certain algebraic condition is satisfied by the variables of interest, the geometric reformulation shows that it suffices to know that a given geometric relationship holds. This is an instance of a much larger motif between algebraic and geometric expressive means that runs throughout various parts of mathematics. The interpretation of scattering amplitudes in terms of the volume of the amplituhedron takes this geometric reformulation even further. It shows that the equivalence of various representations of scattering amplitudes (derived from the BCFW recursion relations using different choices of line-shifts) is no coincidence, since they all correspond to different ways of triangulating a mathematical object known as the amplituhedron.

Insofar as physicists have an epistemic reason to trivialize certain conservation properties, they have an epistemic reason to privilege variables that do so. Imagine then that a physicist judges momentum twistor variables to be more fundamental than spinor–helicity variables. Rather than construing this judgment as involving a metaphysical commitment to joints in nature, we can apply the expressivist analysis from Section 5.6. In making this judgment of relative fundamentality, we implicitly endorse a set of norms on which one ought to prefer variables that can perform the various functional roles that momentum twistors perform (but that spinor–helicity variables cannot). The same could be said for viewing the dual coordinates as being more fundamental than spinor–helicity variables, since the dual coordinates make manifest the conservation of momentum.

## 5.10 Conclusion

We have seen that by changing variables, we can make otherwise obscured or hidden properties manifest. Against instrumentalism, I have argued that good variable choices can have non-instrumental epistemic value. Yet, the challenge of accounting for this epistemic value initially seems to favor fundamentalism. Here, I have shown that conceptualism has ample resources to accommodate the intellectual significance of making properties manifest. Good variable choices make intelligible properties of expressions and patterns in calculations. By changing variables, we sometimes make available new problem-solving plans, with concomitant differences in EDRs.

Sections 5.2-5.5 provided numerous elementary examples of making properties manifest. I showed how Cartesian coordinates make manifest the invariant degrees of freedom of horizontal and vertical lines. Likewise, polar coordinates make manifest properties of circles and diagonal lines. I provided a structurally similar illustration in the context of translating between natural languages.

Section 5.6 considered a rebuttal on behalf of fundamentalism. Scientists and mathematicians frequently judge one choice of variables to be more fundamental than another, especially in the context of making properties manifest. Hence, there is a burden on conceptualism to provide a non-metaphysical account of these practice-based judgments of fundamentality. Using expressivism, I provided one way to discharge this burden. To judge that a variable choice X is more fundamental than a variable choice Y is to express an attitude of *being for privileging* X over Y. If we focus on non-metaphysical reasons for privileging one variable choice over another, then this provides a non-metaphysically committal account of fundamentality.

Finally, Sections 5.8-5.9 developed two case studies of making a hidden symmetry manifest, concerning the hydrogen atom and supersymmetric Yang–Mills theory, respectively. In both cases, making the symmetry manifest requires transforming to new variables. Particularly in the case of supersymmetric Yang–Mills theory, we saw that one can use these new variables to construct objects that are manifestly invariant under the previously hidden symmetry. These manifestly-invariant objects can then be used to construct others, which inherit the property of being manifestly-invariant. Section 5.9 also illustrated numerous epistemic reasons that motivate physicists to transform variables, such as a desire to trivialize conservation of momentum.

Overall, conceptualism provides a promising account of the value of making symmetries manifest and of good variable choices generally. The success of conceptualism in this regard further undermines fundamentalism. Intuitively, manifest symmetries seem like a case where fundamentalism starts out with the upper hand. By showing that we can avoid metaphysically-committal notions of fundamentality *even in these cases*, we gain further reason to believe that we can avoid such commitments generally. If fundamentalism is not needed to account for the non-instrumental value of making properties manifest, it is hard to see where fundamentalism is needed—at least when it comes to assessing the value of compatible reformulations. Consequently, the arguments in this chapter insulate conceptualism from one of the strongest objections that a fundamentalist might leverage against it. Indeed, insofar as fundamentalists typically endorse Occam's razor, they should value the ontological parsimony of my conceptualist account.[31]

[31]Even if one views facts about metaphysical structure as part of a theory's ideology, rather than its ontology, Sider still advocates parsimony considerations here as well (2011, p. 14).